

# Spaces of Particles on Manifolds and Generalized Poincaré Dualities \*

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## §1 Introduction and Statement of Results

There are interesting results throughout the literature relating multi-configuration spaces to mapping spaces (cf. [B], [G], [Gu1-2], [McD], [S1-2]). In this paper, we use a “local to global” *scanning* process based on a construction of Segal to unify and generalize these results.

First of all by a *configuration* on a space  $X$  we mean a collection of unordered points on  $X$  (they can be distinct or not). A multi-configuration will then mean a tuple of configurations with (possibly) certain relations between them. Of course, more rigorous definitions are to follow.

It is known by classical work of G. Segal [S1], that the space of configurations of *distinct* points in Euclidean space is equivalent in homology to an iterated loop space on a sphere. Later work of D. McDuff extended this result to an arbitrary smooth compact manifold (with boundary) where she showed that the space of configurations of distinct points there is equivalent in homology to a space of sections of an appropriate bundle. A bit more later, F. Cohen and C.F. Bodigheimer proved a similar result for spaces of configurations of distinct points *with labels* (see [B]).

Both Segal and McDuff extended their ideas to spaces made out of pairs of configurations. While Segal worked with *divisor spaces* made out of pairs of configurations having no points in common on a punctured Riemann surface [S2], McDuff dealt with what she coined the *space of positive and negative particles* on a general smooth manifold. Both were able to identify these spaces with some function spaces.

This paper extends and generalizes the work of Segal and McDuff in a great many directions. It also sets a context in which these types of results can be viewed and interpreted by relating them to more classical aspects of algebraic topology, as well as to some recent problems stemming from Gauge theory and dealing with the topology of holomorphic mapping spaces.

A starting point for us has been to address the following question: which (multi-) configuration spaces can be used to model mapping spaces (and vice-versa). Such considerations have led us to

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introduce a new and general class of multi-configurations, the *particle spaces* and these include symmetric and truncated products, divisor spaces, configurations of distinct points, spaces of positive and negative particles, and most other known examples in the literature.

Our basic definition is: *A particle space is a multi-configuration space with a partial monoid structure.* If one defines the support of a multi-configuration to mean the locus of the points making up the multi-configuration, then the partial monoid structure will be “concatenation” defined on multiconfigurations having disjoint support.

Given a manifold  $M$ , the most basic example of a particle space on  $M$  is the infinite symmetric product  $SP^\infty(M) = \coprod SP^n(M)$  (and this is a monoid). Another standard particle space is the (traditional) configuration space  $C^\infty(M) \subset SP^\infty(M)$  consisting of unordered disjoint points of  $M$  (and inheriting a partial monoid structure). We agree on the following notation: an element  $\zeta$  in  $SP^n(M)$  can be written both as the formal sum  $\sum n_i x_i, x_i \in M, n_i \in \mathbf{N}$  and  $\sum n_i = n$ , or as an unordered tuple  $\langle x_1, \dots, x_n \rangle$ .

Generally we define  $Par^\infty(M)$  to be any quotient of any subset of a finite product  $\prod SP^\infty(M)$  satisfying the partial monoid structure referred to above. A point of  $Par^\infty(M)$  will then be a multiconfiguration on  $M$  with certain constraints. The particle spaces are defined on any manifold  $M$  (which we do assume in this paper to be smooth) and hence we can talk of a *particle functor*  $Par^\infty$ . Here are some examples of those functors and spaces we study in this paper:

- Symmetric product spaces with “bounded multiplicity”. Given  $M$  as above and an integer  $d \geq 1$ , we define

$$SP_d^\infty(M) = \{ \sum n_i x_i \in SP^\infty(M) \mid n_i \leq d \}$$

Of course  $SP_1^\infty = C^\infty$  is the configuration space of distinct points.

- $Par^\infty(M) = \{ (\zeta_1, \dots, \zeta_k) \in SP^\infty(M)^k \mid \zeta_i \cap \zeta_j = \emptyset, i \neq j \}$ . A related space will be the set of  $k$ -tuples of configurations  $j$  of which are distinct,  $j \leq k$ .
- $Par^\infty(M) = \prod^n SP^\infty(M) / \Delta(SP^\infty(M))$  where  $\Delta$  is the submonoid generated by diagonal elements.
- Truncated symmetric products and these refer to  $TP_p^\infty(M) = SP^\infty(M) / x^p \sim *$  (here we’re thinking of  $SP^\infty(M)$  as a topological monoid with  $*$   $\in M$  the identity element).
- Spaces of positive and negative particles of McDuff and these refer to  $Par^\infty(M) = C^\pm(M) = C(M) \times C(M) / \sim$  where  $\sim$  is the identification

$$(\zeta_1, \eta_1) \sim_R (\zeta_2, \eta_2) \Leftrightarrow \zeta_1 - \eta_1 = \zeta_2 - \eta_2.$$

- The divisor spaces of Segal studied in connection with the space of holomorphic maps of Riemann surfaces into projective spaces (see [K1] and [C<sup>2</sup>M<sup>2</sup>]). They are defined as

$$\text{Div}^n(M) = \{ (\zeta_1, \dots, \zeta_n) \in SP^\infty(M)^n \mid \zeta_1 \cap \zeta_2 \cap \dots \cap \zeta_n = \emptyset \}.$$

The key property of the particle spaces is that when you look closely at a multiconfiguration of  $Par^\infty(M)$  in the neighborhood  $D$  of a point (that is when you *scan* the manifold), what you see is a multiconfiguration living in  $Par^\infty(D)$ . This restriction property turns out to be a direct consequence of the partial monoid structure put on  $Par^\infty(M)$ .

As is standard, one can define relative particle spaces whereby the functor  $Par^\infty$  can be applied to a pair of spaces. If  $N \subset M$ , then  $Par^\infty(M, N)$  consists (roughly) of all those multiconfigurations in  $Par^\infty(M - N)$  which get identified as they approach  $N$ . It is not hard to see that the scanning property mentioned in the previous paragraph establishes (at least for parallelizable manifolds  $M$ ) the existence of a map

$$S : Par^\infty(M) \longrightarrow \text{Map}(M, Par^\infty(S^n, *))$$

where  $*$   $\in S^n$  can be chosen to be the north pole.

For a given space  $M$ ,  $Par^\infty(M)$  is a disconnected partial monoid (with components not very comparable.) It turns out that by “group completing” with respect to this partial monoid structure, one obtains a space  $Par(M)$  which is better behaved (and all of whose components are homeomorphic). The functor  $Par$  (which we construct in §7) is the last ingredient we need and we are now in a position to state the main result of this paper.

**Main Theorem 1.1:** *Let  $M$  be an  $n$  dimensional, smooth, compact (possibly with boundary) and connected manifold. Then there is a fibre bundle*

$$1.2 \quad Par^\infty(S^n, *) \longrightarrow E_{Par^\infty} \longrightarrow M$$

with a (zero) section. Choose  $N$  to be a closed ANR in  $M$  and assume that either  $N \neq \emptyset$  or  $\partial M \neq \emptyset$ . Then there is a homology equivalence (induced by scanning)

$$S_* : H_*(Par(M - N); \mathbf{Z}) \xrightarrow{\cong} H_*(Sec(M, N \cup \partial M, Par^\infty(S^n, *)); \mathbf{Z})$$

where  $Sec(M, A, Par^\infty(S^n, *))$  is the space of sections of 1.2 trivial over  $A$ .

The above theorem has several variants described throughout this paper. An immediate question one asks is when can the homology equivalence of theorem 1.1 be upgraded to a homotopy equivalence. We resolve this as follows.

**Theorem 1.3:** *Let  $N, M$  be as in 1.1 and suppose  $\pi_1(Par(\mathbf{R}^n))$  is abelian, then scanning is a homotopy equivalence*

$$Par(M - N) \xrightarrow{\simeq} Sec(M, N \cup \partial M, Par^\infty(S^n, *)).$$

### Corollaries and Examples:

- When  $M$  is parallelizable, the bundle of configurations 1.2 trivializes and sections turn into maps into the fiber. One therefore has the equivalence

$$H_*(Par(M - N); \mathbf{Z}) \xrightarrow{\cong} H_*(\text{Map}(M, N, Par^\infty(S^n, *)); \mathbf{Z})$$

where  $\text{Map}(M, N, Par^\infty(S^n, *))$  is the space of (based) maps sending  $N$  to the canonical basepoint in  $Par^\infty(S^n, *)$ . When  $N = *$ , we write  $\text{Map}^*(M, Par^\infty(S^n, *))$  for the corresponding mapping space.

- (Segal [S1]) Let  $M_g$  be a genus  $g$  Riemann surface. Then  $\text{Div}^2(M_g - *) \simeq \text{Map}_c^*(M, \mathbf{P} \vee \mathbf{P})$  where  $\mathbf{P} = K(\mathbf{Z}, 2)$  is the infinite complex projective space and where  $\text{Map}_c^*$  is any component of the subspace of based maps (see 11.4).
- (McDuff [McD1])  $C^\pm(\mathbf{R}^n) \simeq \Omega^n(S^n \times S^n / \Delta)$  where  $\Delta$  is the diagonal copy of  $S^n$  in  $S^n \times S^n$ .
- Let  $C$  be the configuration functor associated to  $C^\infty$ . Then

$$1.4. \quad H_*(C(\mathbf{R}^n); \mathbf{Z}) \xrightarrow{\cong} H_*(\Omega^n(S^n); \mathbf{Z}) \quad [\text{S2}]$$

- We can generalize 1.4 as follows. Let  $C^{(k)}(\mathbf{R}^n) \subset \prod^k C(\mathbf{R}^n)$  consist of the subspace of *pairwise disjoint* configurations. Then

$$H_*(C^{(k)}(\mathbf{R}^n); \mathbf{Z}) \xrightarrow{\cong} H_*(\Omega^n(\underbrace{S^n \vee \cdots \vee S^n}_k); \mathbf{Z}).$$

Other interesting examples we discuss are the symmetric products with bounded multiplicity which we introduced earlier and denoted by  $SP_d^\infty(M)$ ,  $d \geq 1$ . For  $M$  either open or with boundary, the “partial” completion  $SP_d(M)$  has the following very simple description. Choose an end (or a tubular neighborhood of the boundary) and construct a nested sequence  $\{U_i\}$  of neighborhoods of it. By choosing a sequence of disjoint points  $z_i \in U_i - U_{i+1}$  we obtain maps

$$SP_d^\infty(M - U_i) \xrightarrow{+z_i} SP_d^\infty(M - U_{i+1}) \xrightarrow{+z_{i+1}} \cdots$$

and the direct limit we denote by  $SP_d(M)$ . The following proposition, which we state in the special case  $M = \mathbf{R}^n$  (see theorem 11.7), is a direct corollary of 1.1 and 1.3 once we observe that  $SP_d^\infty(S^n, *) \simeq SP^d(S^n)$  (cf. §11).

**Proposition 1.5:** *Scanning  $S$  is a homotopy equivalence*

$$SP_d(\mathbf{R}^n) \xrightarrow{\simeq} \Omega_0^n SP^d(S^n)$$

*whenever  $d > 1$ , and a homology equivalence when  $d = 1$ .*

**Note:** The proposition above has also been obtained by M. Guest, A. Kozłowski and K. Yamaguchi [GKY] (who state it for the case  $n = 2$ ; cf. §11.2). A labelled analog of it is given in [K2] and yields a direct generalization of the May-Milgram model for iterated loop spaces. One might note that 1.5 provides yet another extension of Segal’s result (1.4).

One main interest in theorem 1.1 is the way it relates to and generalizes many of the classical dualities on manifolds. The following theorem (obtained earlier by Pawel Gajer [G] using different techniques) is obtained after a close analysis of the bundle 1.2 for the case  $Par^\infty = SP^\infty$ .

**Theorem 1.6:** *Let  $M$  be  $n$  dimensional, smooth and compact, and let  $N$  be an ANR in  $M$ . Suppose  $M$  is orientable. Then scanning induces a homotopy equivalence*

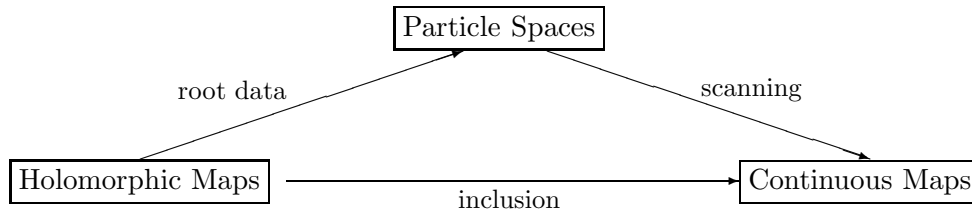
$$S : SP^\infty(M - N, *) \xrightarrow{\simeq} \text{Map}_c(M, N \cup \partial M, SP^\infty(S^n, *))$$

*where  $\text{Map}_c$  is any component of the space of (based) maps.*

**Corollary 1.7** (Alexander-Lefschetz-Poincaré): *Let  $M$  and  $N$  be as above, then*

$$\tilde{H}_*(M - N; \mathbf{Z}) \cong H^{n-*}(M, N \cup \partial M, \mathbf{Z}).$$

This work finds its origins in an attempt to construct *configuration space models* for spaces of holomorphic maps on Riemann surfaces  $M_g$ . In the past decade and as a result of the increasing “*rapprochement*” between mathematics and physics, there has been a flurry of activity towards understanding the topology of spaces  $\text{Hol}^*(M_g, X)$  of (based) holomorphic functions into various algebraic varieties. The general picture that emerges there is that for many special rational  $X$ ’s one has the following relationships (eg. [Gu1-2], [BHMM], [KM])



In this framework, one uses the particle spaces to provide models for spaces of holomorphic maps on a Riemann surface, which themselves are *suitable approximations* to spaces of all maps. The following (perhaps unsuspected) corollary is given in §14 and it recuperates a known theorem of M. Guest [Gu2].

**Corollary 1.8:** (Guest) *Let  $X$  be a projective toric variety (non-singular). The natural inclusions  $i_D : \text{Hol}_D^*(S^2, V) \longrightarrow \Omega_D^2 V$  (where  $D$  are multidegrees depending on  $V$ ) induce a homotopy equivalence when  $D$  goes to  $\infty$ ; i.e.*

$$\lim_{D \rightarrow \infty} \text{Hol}_D^*(S^2, V) \xrightarrow{\simeq} \Omega_0^2 V$$

where  $\Omega_0^2 V$  is any component of  $\Omega^2 V$ .

**REMARK 1.9:** The equivalence above between spaces of rational maps and loop spaces of certain projective varieties has been observed initially by Segal for the case of  $V = \mathbf{P}^n$  and later extended to more general flag manifolds by several authors (see [C<sup>2</sup>M<sup>2</sup>], [BHMM] and references therein). In light of the methods used in this paper, it turns out that it is precisely the partial monoid structure that is exhibited by the root data of rational maps on toric varieties that induces the equivalence with the second loop space of  $V$ . This shouldn’t be surprising in light of earlier work of Segal ([S4]) and should provide an interesting insight into why equivalences of the sort should hold.

Finally, it is not hard to see that the ideas presented above apply equally well (but in a different context) to obtaining space level descriptions of Spanier-Whitehead duality for any generalized homology theory (cf. §15).

**Theorem 1.10:** Let  $E$  be a connected  $\Omega$  spectrum and define the functor  $F_{\mathbf{E}}(-) = \Omega^\infty(\mathbf{E} \wedge -)$  on the category of CW complexes. Then for all  $X \in CW$ , there is a homotopy equivalence

$$S : F(X) \xrightarrow{\simeq} \text{Map}_*(D(X, k), F(S^k))$$

where  $D(X, k) = S^k - X$  is the Spanier-Whitehead dual of  $X \hookrightarrow S^k$ .

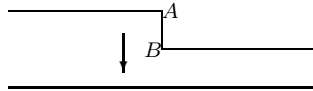
**Corollary 1.11:** (Spanier-Whitehead duality) Let  $h$  be any homology theory and suppose  $A, B \in S^k$ ,  $A$  and  $B$  are  $n$  dual. Then there is an isomorphism

$$h_i(B) \cong h^{n-1-i}(A).$$

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## §2 Quasifibrations and Homology Fibrations

DEFINITION: Recall a map  $f : Y \rightarrow X$  is a quasifibration if  $\forall x \in X$  the inclusion of  $f^{-1}(x)$  into the homotopy fiber over  $x$  is a weak homotopy equivalence. Roughly speaking, while a fibration enjoys the property of homotopy lifting “on the nose”, one may need to “deform” homotopies before being able to lift them for the case of a quasifibration. A standard example is given by the projection  $\pi$  depicted below



Clearly,  $\pi$  is not a fibration for one cannot lift homotopies that don’t “spend much time” at the point  $\{\frac{1}{2}\} = \pi([AB])$ . This projection is however a quasifibration and by allowing homotopies to “live a while” over certain closed sets ( $\{\frac{1}{2}\}$  in this case) one should be able to lift them. This is the essence of 2.2 below.

DEFINITION 2.1: A map  $\pi : E \rightarrow B$  is a homology fibration if for each  $b \in B$ , the natural map  $\pi^{-1}(b) \rightarrow F$  into the homotopy fiber is a homology equivalence.

The general criterion developed by Dold and Thom in [DT] to show that a map is a quasifibration can be extended to include the case of homology fibrations as well. This gives

**Criterion 2.2:** [DT] Let  $X_1 \subset \cdots \subset X_k \subset \cdots$  be a (finite) filtration of  $X$  by closed subspaces and let  $f : Y \longrightarrow X$  be a map satisfying

(i)  $f$  is a fibre bundle over  $X_{k+1} - X_k$  with fiber  $F$

(ii) There is an open set  $X_{k-1} \subset U_k \subset X_k$  and a deformation retraction of  $r_t$  of  $U_k$  to  $X_{k-1}$  which can be lifted to a deformation retraction  $\tilde{r}_t$  (upstairs) of  $f^{-1}(U_k)$  to  $f^{-1}(X_{k-1})$ .

The map  $f$  is a quasifibration (resp. homology fibration) if  $\tilde{r}_1 : f^{-1}(x) \rightarrow f^{-1}(r_1(x))$  is a weak homotopy (resp. homology) equivalence for all  $x \in U_k$ .

TERMINOLOGY: The maps  $\tilde{r}_1$  are referred to by McDuff as *attaching maps*. We will adopt the same terminology.

**Remark 2.3:** A slightly more general version of 2.2 holds: Suppose  $X_{\vec{k}} = X_{k_1, \dots, k_n}$  is a cover of  $X$  by closed subsets such that  $X_{k_1, \dots, k_n} \subset X_{k_1, \dots, k_{i+1}, \dots, k_n}$  for all  $1 \leq i \leq n$  and suppose that  $\pi$  is a trivial fibration over  $X_{k_1, \dots, k_n} - \bigcup_i X_{k_1, \dots, k_{i-1}, \dots, k_n}$ . Let  $U_{\vec{k}}$  and the attaching maps  $\tilde{r}_1$  be defined as in 2.2 where  $U_{\vec{k}}$  retracts down to  $X_{\vec{k}}$  via a retraction  $r$ . The very same criterion as in 2.2 states that if the attaching maps over the  $X_{\vec{k}}$  are homotopy (resp. homology) equivalences, then  $\pi$  is a quasifibration (resp. a homology fibration).

EXAMPLE 2.4: We illustrate 2.2 for the projection  $\pi : Y \rightarrow [0, 1]$  depicted earlier. Set  $X_1 = \frac{1}{2}$  and  $X_2 = [0, 1]$ . One can let  $U_1$  be  $(0, 1)$ . The retraction  $r$  over the time interval  $[0, 2]$  is chosen to shrink linearly  $(0, 1)$  to  $\{\frac{1}{2}\}$  over the time interval  $[0, 1]$  and to be stationary at  $\frac{1}{2}$  for  $t \in [1, 2]$ . Now  $\tilde{r}_1$  corresponds to the following: it shrinks  $Y$  to the vertical line segment  $AB$  (linearly over the time interval  $[0, 1]$ ) leaving  $AB$  fixed. Then for  $t \in [1, 2]$ , it slides the point  $A$  to the end point  $B$  along  $AB$ . By construction, this gives a lift  $\tilde{r}$  of  $r$  and we have that  $\tilde{r}_1(r^{-1}(U_1)) = \{B\}$ .

EXAMPLE 2.5: We defined earlier  $SP^\infty(M) = \coprod SP^n(M)$ . Choose a basepoint  $* \in M$  and construct inclusions  $SP^n(M) \hookrightarrow SP^{n+1}(M)$  given by adjoining basepoint  $\sum n_i x_i \mapsto \sim n_i x_i + *$ . The direct limit is denoted by  $SP^\infty(M, *)$ . The following classical theorem of Dold-Thom will serve as a prototype for later proofs.

DEFINITION 2.6: For  $* \in N \subset M$ , we define  $SP^\infty(M, N)$  to be  $SP^\infty(M/N, *)$ . An equivalent description of this space is given in §6.

**Proposition 2.7:** Let  $N \hookrightarrow M \rightarrow M/N$  be a cofibration and choose a basepoint  $* \in N \subset M$ . Then

$$SP^\infty(N, *) \longrightarrow SP^\infty(M, *) \longrightarrow SP^\infty(M, N)$$

is a quasifibration.

PROOF: Let  $X^k = SP^k(M, N)$  be the image of  $SP^k(M)$  under the quotient map  $SP^k(M) \rightarrow SP^k(M/N) \hookrightarrow SP^\infty(M, N)$ . It should be clear that

$$X_k = \{D \in SP^\infty(M, N) \mid \text{card}(D \cap (M - N)) \leq k\}$$

and that the  $X_k$  provide an increasing filtration of the  $SP^\infty(M, N)$ . Since  $N \hookrightarrow M$  is a cofibration, there is a neighborhood retract  $U$  of  $N$  in  $M$ ; that is there is an open  $N \subset U \subset M$  and a continuous  $r : M \longrightarrow M$  such that  $r$  leaves  $M - U$  and  $N$  invariant and maps  $U$  to  $N$ . The map  $r$

lifts (additively) to  $SP^\infty(M)$  and we write this map as  $\tilde{r}$ . Let now

$$U_k = \{D \in SP^\infty(M, N) \mid \text{card}(D \cap (M - N)) \leq k \text{ and at least one element of } D \text{ is in } U\}.$$

Clearly  $X_{k-1} \subset U_k \subset X_k$  and  $r|_{U_k}$  retracts  $U_k$  onto  $X_{k-1}$ . It is also clear that over

$$X_k - X_{k-1} = \{D \in SP^\infty(M, N) \mid \text{card}(D \cap (M - N)) = k\}$$

the projection  $\pi : SP^\infty(M) \rightarrow SP^\infty(M, N)$  is a product  $\pi^{-1}(X_k - X_{k-1}) = SP^\infty(N) \times (X_k - X_{k-1})$  and hence is trivial there. It remains then to check condition (ii) of 2.2.

Let  $b = \langle z_1, \dots, z_k \rangle \in X_k$  and  $z_i \in M - N$ . Then  $b \in U_k$  if one of the  $z_i$  is in  $U$ . Write  $b = \langle z_1, \dots, z'_1, \dots \rangle$  where  $z_i \in M - U$  and  $z'_i \in U - N$ . Let  $F$  denote  $SP^\infty(N)$ . One uses the trivialization of  $\pi$  over  $X_k - X_{k-1}$  to write  $\pi^{-1}(b) = b + F$  and so the lifted retraction  $\tilde{r} : \pi^{-1}(b) \longrightarrow \pi^{-1}(r(b))$  takes the form

$$\begin{aligned} \tilde{r}_1 : \langle z_1, \dots, z'_1, \dots \rangle + F &\longrightarrow \langle r(z_1), \dots, r(z'_1), \dots \rangle + F \\ &\longrightarrow \langle r(z_1), \dots \rangle + (\langle r(z'_1), \dots \rangle + F) \end{aligned}$$

But since  $r_1(z'_i) \in N, \forall i$ , they can be connected to basepoint by paths and this defines a homotopy of  $(\langle r(z'_1), \dots \rangle + F) \simeq F$  and hence  $\tilde{r}_1(\pi^{-1}(b)) \simeq \langle r_1(z_1), \dots \rangle + F \simeq \pi^{-1}(r_1(b))$ .  $\blacksquare$

### §3 Particle Functors and Particle Spaces

In this section, we define the  $Par^\infty$  spaces associated to path connected spaces  $M$ . The starting point here is the multiconfiguration space  $\prod^k SP^\infty(M)$ . We are interested in subsets and quotients of this space satisfying an “adjunction” condition.

TERMINOLOGY: Consider the  $n$ -tuple of configurations  $\vec{\zeta} = (\zeta_1, \dots, \zeta_k) \in SP^\infty(M)^{\times n}$ .

- The support of  $\zeta_i$  is the set of points making up  $\zeta_i$  and the support of  $\vec{\zeta}$  is the union of the supports of the  $\zeta_i$ .
- A *subtuple*  $\vec{\zeta}'$  of  $\vec{\zeta}$  consists of a  $k$ -tuple  $(\zeta'_1, \dots, \zeta'_k)$  of subconfigurations  $\zeta'_i \subset \zeta_i$ .
- $\vec{\zeta}$  is said to lie in  $A \subset M$  if the support of  $\vec{\zeta}$  is in  $A$ . Equivalently  $\vec{\zeta} \in SP^\infty(A)^k \subset SP^\infty(M)^k$ .
- Two configurations  $\vec{\zeta}$  and  $\vec{\zeta}'$  are *distinct* if their supports are distincts. They are *disjoint* if they lie in disjoint subsets of  $M$ ; i.e. if  $\vec{\zeta} \cap \vec{\zeta}' = \emptyset$ . Of course disjoint implies distinct.
- Given  $A \subset SP^\infty(M)^k$  and  $\vec{\zeta} \in SP^\infty(M)^k$  then  $\vec{\zeta} \cap A$  is the subtuple of  $\vec{\zeta}$  made out of the points of  $\vec{\zeta}$  that are in  $A$ .

DEFINITION AND NOTATION:  $\prod SP^\infty(M) = SP^\infty(M)^k$  is a topological monoid and we write its pairing as  $+$ . We denote by  $\underline{\mathcal{C}}$  be the category of spaces with *injections* as morphisms.

DEFINITION 3.1 (**Particle Spaces of the first kind**): A (sub) *particle* functor  $Par^\infty$ , or a particle space of the *first kind*, is a *covariant* functor  $\underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$  satisfying the following two properties:

**P1**  $Par^\infty : M \mapsto Par^\infty(M) \subset SP^\infty(M)^k, \text{ for some } k > 0, \forall M \in \underline{\mathcal{C}}$

and  $\forall A, B \subset M \in \underline{\mathcal{C}}, A \cap B = \emptyset$ , the symmetric product pairing  $+$  yields an identification

**P2**  $Par^\infty(A \sqcup B) = Par^\infty(A) + Par^\infty(B) \subset Par^\infty(M).$



**Remark 3.2:** The symmetric product pairing  $+$  restricts to a partial pairing on  $Par^\infty$  whereby element can be added only if they have disjoint support. This endows  $Par^\infty(M)$  with a *partial monoid* structure. Note that the functoriality of  $Par^\infty$  implies the following naturality for all  $N \subset M$

$$\begin{array}{ccc} Par^\infty(N) & \hookrightarrow & Par^\infty(M) \\ \downarrow \subset & & \downarrow \subset \\ SP^\infty(N)^k & \hookrightarrow & SP^\infty(M)^k. \end{array}$$

**Example 3.3:** Let  $A \subset M \in \underline{\mathcal{C}}$ ,  $A$  open, and let  $F(M)$  be the space

$$F(M) = \left\{ (\zeta_1, \zeta_2, \zeta_3) \in SP^\infty(M)^3 \mid \zeta_1 \cap A = \emptyset. \right\}$$

$F(M)$  is not a particle space since it is not induced from a functor.

**Example 3.4:** Consider the space  $F(M) \subset SP^\infty(M)^3$  consisting of triples  $(\zeta_1, \zeta_2, \zeta_3)$  such that

$$\deg(\zeta_1) = \deg(\zeta_2) + \deg(\zeta_3).$$

Then  $F$  defines a functor  $\underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$ . It however doesn't satisfy **P2** for it is easy to see that the inclusion  $F(A) + F(B) \subset F(A \sqcup B)$  is proper.

The following gives a description of particle functors of the first kind.

**Lemma 3.5:** Let  $F$  be a functor  $\underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$  satisfying **P1**; that is  $\exists k > 0$  such that  $F(M) \subset SP^\infty(M)^k$  for all  $M \in \underline{\mathcal{C}}$ . Then  $F$  is a  $Par^\infty$  functor if and only if for all  $N \subset M$

$$F(N) = F(M) \cap SP^\infty(N)^k.$$

PROOF: Let  $A$  and  $B$  be disjoint in  $M$  and suppose  $F(A) = F(M) \cap SP^\infty(A)^k$  (same for  $B$ ). Then

$$\begin{aligned} F(A \sqcup B) &= F(A \sqcup B) \cap SP^\infty(A \sqcup B)^k \\ &= F(A \sqcup B) \cap (SP^\infty(A)^k \times SP^\infty(B)^k) = F(A \sqcup B) \cap (SP^\infty(A)^k + SP^\infty(B)^k) \\ &= F(A \sqcup B) \cap SP^\infty(A)^k + F(A \sqcup B) \cap SP^\infty(B)^k \\ &= F(A) + F(B) \end{aligned}$$

and  $F$  is indeed a  $Par^\infty$  functor. Suppose now that  $F = Par^\infty$  for some particle functor and let  $N \subset M \in \underline{\mathcal{C}}$ . Then

$$\begin{aligned} Par^\infty(M) \cap SP^\infty(N)^k &= (Par^\infty(N) + Par^\infty(M - N)) \cap SP^\infty(N)^k \\ &= Par^\infty(N) \cap SP^\infty(N)^k + Par^\infty(M - N) \cap SP^\infty(N)^k \\ &= Par^\infty(N) \cap SP^\infty(N)^k = Par^\infty(N) \end{aligned}$$

as desired and this proves the lemma. ■

**DEFINITION 3.6 (Particle functors):** A functor  $Par^\infty : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$  is a particle functor if there are quotient maps

$$\mathbf{Q1} \quad q_M : Par_1^\infty(M) \longrightarrow Par^\infty(M), \quad \forall M \in \underline{\mathcal{C}}$$

of some particle space of the first kind such that the partial pairing  $+$  on  $Par_1^\infty$  descends to a partial pairing of quotient spaces; i.e.

$$\begin{array}{ccc} Par_1^\infty(A) \times Par_1^\infty(B) & \xrightarrow{+} & Par_1^\infty(M) \\ \downarrow q_A \times q_B & & \downarrow q_M \\ Par^\infty(A) \times Par^\infty(B) & \xrightarrow{+} & Par^\infty(M) \end{array}$$

whenever  $A, B \subset M$ ,  $A \cap B = \emptyset$ .

**Remark 3.7:** When  $Par^\infty$  is a quotient of  $\coprod SP^\infty$ , then  $A$  and  $B$  don't need to be distinct and we can demand that  $+$  commutes with  $q$  i.e

$$\begin{array}{ccc} SP^\infty(M)^k \times SP^\infty(M)^k & \xrightarrow{+} & SP^\infty(M)^k \\ \downarrow q \times q & & \downarrow q \\ Par^\infty(M) \times Par^\infty(M) & \xrightarrow{+} & Par^\infty(M) \end{array}$$

and so  $Par^\infty(M)$  in this case has automatically a monoidal structure given by  $+$  above.

**NOTATION:** We write an element  $\zeta \in Par^\infty(M)$  as a tuple  $(\zeta_1, \dots, \zeta_n)$  which could either be in  $SP^\infty(M)^k$  or could represent  $q^{-1}(\zeta)$  in  $SP^\infty(M)^k$ .

**Remark 3.8:** It has been pointed out to the author that definition 3.6 is closely related to a similar definition of M. Weiss dealing with spaces of immersions and embeddings of manifolds (eg. preprint “Embeddings from the point of view of Immersion theory”). Unfortunately we are not very knowledgeable of the work of Weiss at this point to make the analogy precise.

## §4 Construction of Particle Spaces

**Definition 4.1:** Let  $U$  be a topological (partial) monoid and  $A \subset U$  any subspace. Then by  $U//A$  we mean the identification space

$$U//A = U/a + x \sim x, \quad a \in A \text{ and whenever } a + x \text{ is defined.}$$

If  $U$  is a monoid and  $A$  a submonoid, then  $U//A$  is simply the quotient monoid.

**Lemma 4.2:** Let  $Par_1^\infty$  and  $Par_2^\infty$  be two particle functors,  $M \in \underline{\mathcal{C}}$ .

- (a) Then  $Par_1^\infty(M) \times Par_2^\infty(M)$  is a particle space and  $Par_1^\infty \times Par_2^\infty$  a particle functor.
- (b) If  $Par_1^\infty(M) \subset Par_2^\infty(M)$ , then  $Par_1^\infty(M)//Par_2^\infty(M)$  is a particle space.

PROOF: We observe that

$$\begin{aligned} Par_1^\infty(A \sqcup B)^k // Par_2^\infty(A \sqcup B) &= [Par_1^\infty(A)^k + Par_1^\infty(B)^k] // [Par_2^\infty(A) + Par_2^\infty(B)] \\ &= Par_1^\infty(A)^k // Par_2^\infty(A) + Par_1^\infty(B) // Par_2^\infty(B) \end{aligned}$$

and (b) follows. ■

EXAMPLE 4.3: A space we looked at before is  $C^\pm$  which is given as the quotient of  $C^\infty(M)^2 \subset SP^\infty(M)^2$  by  $Par_2^\infty(M) = \Delta C^\infty(M)$  where  $\Delta$  is the diagonal

$$\Delta : C^\infty(M) \longrightarrow C^\infty(M) \times C^\infty(M), \quad \zeta \mapsto (\zeta, \zeta).$$

Consider now for each  $M \in \underline{\mathcal{C}}$  a map of monoids  $f_M : SP^\infty(M)^m \longrightarrow SP^\infty(M)^n$ ,  $m, n$  positive integers. We assume that the maps  $f_M, M \in \underline{\mathcal{C}}$  are compatible with inclusions  $N \subset M$ ; that is there are commutative diagrams

$$\begin{array}{ccc} SP^\infty(N)^m & \xrightarrow{f_N} & SP^\infty(N)^n \\ \downarrow \subset & & \downarrow \subset \\ SP^\infty(M)^m & \xrightarrow{f_M} & SP^\infty(M)^n. \end{array} \quad 4.4$$

**Definition 4.5:** Given a subset  $\emptyset \neq A \subset \prod_1^n SP^\infty(M) = SP^\infty(M)^{\times n}$  we denote by  $(A) \in SP^\infty(M)$  the submonoid

$$(A) = \{a + x, a \in A, x \in SP^\infty(M)^{\times n}\}.$$

**Proposition 4.6:** Let  $f_M$  be defined as above for  $M \in \underline{\mathcal{C}}$ . Then both  $Im(f_M) \subset \prod_1^n SP^\infty(M)$  and the complement

$$\prod_1^n SP^\infty(M) - (Im(f_M))$$

are  $Par^\infty$  spaces of the first kind.

PROOF: To verify **P2** for the case  $Par^\infty(M) = Im(f_M)$  notice that for  $A \cap B = \emptyset$  in  $M$ , we have that  $SP^\infty(A \sqcup B) = SP^\infty(A) \times SP^\infty(B)$  and hence  $SP^\infty(A \sqcup B)^m = SP^\infty(A)^n \times SP^\infty(B)^m$ . This then gives

$$\begin{aligned} f_{A \sqcup B}(SP^\infty(A \sqcup B)^m) &= f_{A \sqcup B}(SP^\infty(A)^m \times SP^\infty(B)^m) \\ &= f_A(SP^\infty(A)^m) \times f_B(SP^\infty(B)^m) \end{aligned}$$

and **P2** for this case follows.

Notice at this point that 4.4 implies the existence of a relative map

$$f_{M,N} : \prod_1^m SP^\infty(M, N) \longrightarrow \prod_1^n SP^\infty(M, N).$$

and we can identify  $Par^\infty(M, N)$  with  $(Imf_{M,N})$  in the first case and with its complement in  $SP^\infty(M, N)^{\times n}$  in the second.

We now verify **P2** for spaces of the form  $Par^\infty(M) = \prod^n SP^\infty(M) - (Im(f_M))$ . Let  $A, B \subset M$  as before, then

$$\begin{aligned} & Par^\infty(A) \times Par^\infty(B) \\ &= (SP^\infty(A)^n - (Imf_A)) \times (SP^\infty(B)^n - (Imf_B)) \\ &= SP^\infty(A)^n \times SP^\infty(B)^n - (SP^\infty(A)^n \times Imf_B \cup Imf_A \times SP^\infty(B) \cup Imf_A \times Imf_B) \\ &= SP^\infty(A \sqcup B)^n - (Imf_A \times Imf_B) = SP^\infty(A \sqcup B)^n - (Imf_{A \sqcup B}) \\ &= Par^\infty(A \sqcup B). \end{aligned}$$

The proposition follows. ■

**Corollary 4.7:** Let  $f_M : SP^\infty(M)^n \rightarrow SP^\infty(M)^n$  be as before, then the quotient monoids below form particle spaces

$$Par^\infty(M) = SP^\infty(M)^n // Imf_M \quad \text{and} \quad Par^\infty(M) = SP^\infty(M)^n // (SP^\infty(M)^n - (Imf_M)).$$

EXAMPLE 4.8: Consider the diagonal map

$$M \xrightarrow{\Delta} M \times M \xrightarrow{+} SP^2(M).$$

We extend it multiplicatively to a map  $f_M : SP^\infty(M) \rightarrow SP^\infty(M)$  and it is direct to see that the complement of  $(Imf_M)$  is  $C^\infty(M)$ .

EXAMPLE 4.9: Consider the map

$$M \times M \longrightarrow SP^\infty(M)^{\times 3}, \quad (a, b) \mapsto (a, b, a + b)$$

and extend it additively to a map  $f_M : SP^\infty(M)^{\times 2} \rightarrow SP^\infty(M)^{\times 3}$ . Then  $Imf_M$  corresponds to triples of configurations  $(\zeta_1, \zeta_2, \zeta_3)$  such that  $\zeta_3 = \zeta_1 + \zeta_2$ .

EXAMPLE 4.10: Spaces of pairwise disjoint configurations;  $DDiv^n$  (already defined in the introduction) can be described along the lines formulated above. Assume for example  $n = 3$ , then  $DDiv^3(M)$  is the complement in  $SP^\infty(M)^3$  of  $(Imf_M)$  where  $f_M$  is given by

$$f_M : SP^\infty(M)^3 \longrightarrow SP^\infty(M)^3, \quad (\zeta, \eta, \psi) \mapsto (\zeta + \eta, \zeta + \psi, \eta + \psi).$$

## §5 Some Topological Properties

Naturally  $Par^\infty(M)$  inherits its topology from  $\prod SP^\infty(M)$  and the topology on  $SP^\infty(X)$  is the weak topology relative to the subspaces  $SP^r(X), r \geq 1$ ; that is a set  $U \subset SP^\infty(X)$  is closed if and only if  $U \cap SP^r(X)$  is closed for all  $k$ .

**Lemma 5.1:** *Let  $Par^\infty$  be a particle functor and let  $M$  be a manifold of dimension  $n \geq 1$  such that  $Par^\infty(M) \neq \emptyset$ . Then  $Par^\infty(A) \neq \emptyset$  of all open  $A \subset M$ .*

PROOF: Let  $(\zeta_1, \dots, \zeta_k) \in Par^\infty(M)$ . Then  $S = \{\zeta_1 \cup \dots \cup \zeta_k\}$  is a finite set of points and so there is always an injection of  $\tau : S \rightarrow A$ . Since  $Par^\infty$  is a functor from  $\underline{\mathcal{C}}$  to  $\underline{\mathcal{C}}$ , it follows that there is an induced injection sending  $\{(\zeta_1, \dots, \zeta_k)\} \in Par^\infty(S)$  into  $Par^\infty(A)$  and the lemma follows. ■

Recall that  $Par^\infty$  is a self-functor of the category  $\underline{\mathcal{C}}$  of spaces and injections as morphisms. In particular,  $Par^\infty$  takes inclusions to inclusions. Using the isotopy properties of  $Par^\infty(-)$  the following is not hard to establish.

**Lemma 5.2:** *Let  $M$  be compact with boundary and denote by  $M^{int}$  its interior. Then we have a homeomorphism  $Par^\infty(M) \cong Par^\infty(M^{int})$ .*

DEFINITION 5.3: We let  $\underline{\mathcal{C}}_n \subset \underline{\mathcal{C}}$  consist of the subcategory of  $n$  dimensional ( $n \geq 1$ ), smooth, connected and compact manifolds.

**Isotopy and injective homotopy :** From the functorial properties of  $Par^\infty$ , it is clear that any injective homotopy  $h_t : U \rightarrow M$ ; i.e. a homotopy through injective maps, induces a homotopy of particle spaces;  $Par^\infty(h_t) : Par^\infty(U) \rightarrow Par^\infty(M)$ . An isotopy between  $f, g : N \rightarrow M$  is on the other hand a differentiable homotopy through embeddings. It is ambient if there is an isotopy  $F : M \times I \rightarrow M$  such that  $F(x, 0) = f(x)$  and  $F(x, 1) = g(x)$ .

Given any two points  $p$  and  $q \in int(M)$ ,  $M$  connected, any smooth path between them gives rise to an isotopy from  $p$  to  $q$ . This isotopy can be extended to an ambient isotopy [Ko]. And generally one has

**Lemma 5.4:** *On a manifold  $M \in \underline{\mathcal{C}}_n$ , there is an ambient isotopy taking any finite set of interior points to any other set of interior points with the same cardinality.*

PROOF: Two isotopic embeddings, via an isotopy  $F : N \times I \rightarrow M$ , need not be ambient isotopic. However there is an extension theorem of Thom that gives sufficient conditions for when this is possible; namely when  $N$  is compact and  $M$  is closed. A close inspection of the proof shows that the ambient isotopy can be chosen so that it leaves all points outside a compact neighborhood  $V$  of  $N \times I$  fixed; [Mi].

Therefore and as long as this neighborhood  $V$  misses the boundary of  $M$ , the theorem of Thom still applies for non-closed  $M$ , namely for  $int(M)$ . In our case,  $N$  is a collection of points  $\{x_1, \dots, x_m\} \in int(M)$  and hence is compact. Let  $\{y_1, \dots, y_m\} \in int(M)$  be any other set of  $m$  points and choose paths  $\gamma_i$  between  $x_i$  and  $y_i$  that lie in the interior. Traveling along the paths (at different speeds if need be in order to avoid intersections at any given time) gives an isotopy  $F : N \times I \rightarrow M$ . By Thom's theorem,  $F$  extends to an ambient isotopy and the lemma follows. ■

**Corollary 5.5:** *Let  $N \in \underline{\mathcal{C}}_n$  be a connected space and assume  $Par^\infty(N) \subset SP^\infty(N)^k$ . Then  $Par^\infty(N)$  has  $\mathbf{Z}^+ \times \dots \times \mathbf{Z}^+$  components obtained as the intersection of  $Par^\infty(N)$  with  $SP^{n_1}(N) \times \dots \times SP^{n_k}(N)$  for all tuples of positive integers  $(n_1, \dots, n_k)$ . (Compare 3.5)*

NOTATION: In the case of particle spaces of the first kind, we can then index the components as follows

$$Par_{m_1, \dots, m_k}^\infty(M) = Par^\infty(M) \cap SP^{m_1}(M) \times \dots \times SP^{m_k}(M) \subset \prod_{i=1}^k SP^\infty(M), \quad m_i > 0.$$

More generally, if  $Par^\infty(M)$  is any particle space given as a quotient  $q : Par_1^\infty(M) \rightarrow Par^\infty(M)$  for some first kind  $Par_1^\infty$ , then we define

$$Par_{m_1, \dots, m_k}^\infty(M) = q(Par_{1, m_1, \dots, m_k}^\infty(M)).$$

We will see later (9.17) that the multidegrees  $(m_1, \dots, m_k)$  parametrize maps from  $H_n(M; \mathbf{Z})$  into  $H_n(Par^\infty(S^n, *); \mathbf{Z})$ .

**Lemma 5.6:** *Let  $M \in \mathcal{C}_n$ , and let  $N \subset M$  be an absolute neighborhood retract. Then  $Par^\infty(M, N)$  is connected.*

PROOF:  $N$  being as above, there is an open  $U \subset M$  containing  $N$  and retracting to it via a retraction  $r$ . We assume this retraction is injective on  $N - U$  (think of a collar). Given a multiconfiguration  $\{\zeta_1 \cup \dots \cup \zeta_k\}$  in  $Par^\infty(M, N)$  (see note preceding 3.8), we let its support be the set of points making up the  $\zeta_i$ 's. If this support lies in  $U$ , then the retraction  $r$  takes  $\{\zeta_1 \cup \dots \cup \zeta_k\}$  to  $N$  and hence to basepoint in  $Par^\infty(M, N)$ . Generally if  $\vec{\zeta} = \{\zeta_1 \cup \dots \cup \zeta_k\}$  has support in  $M - N$ , then there always is an isotopy taking  $\vec{\zeta}$  to an element  $\vec{\zeta}'$  in  $U$  (by lemma 5.4). Composing this with  $r$  gives at the end a path connecting  $\{\zeta_1 \cup \dots \cup \zeta_k\}$  to basepoint and the lemma follows. ■

**Example 5.7:** Choose a basepoint  $* \in M \in \mathcal{C}_n$  which is an interior point. Then  $Par^\infty(M, *)$  is connected. We show in §9 that if  $M$  is  $n$  connected then so is  $Par^\infty(M, *)$ .

## §6 Particle Spaces and Cofibrations

**§6.1 Restrictions and Relative Constructions:** Fix a particle functor  $Par^\infty$  and let  $M \in \mathcal{C}$  and  $* \in N \subset M$  closed. Naturally  $SP^\infty(N)$  is a submonoid of  $SP^\infty(M)$  and we define  $SP^\infty(M, N)$  as the quotient monoid  $SP^\infty(M)/SP^\infty(N)$ . When  $N = *$ , it can be checked that this construction agrees with the previously defined  $SP^\infty(M, *)$  in 2.5. Now suppose  $Par^\infty(-)$  is a functor of the first kind, then we define

$$Par^\infty(M, N) = \left\{ \zeta \in SP^\infty(M, N)^k \mid \zeta \cap (M - N) \in Par^\infty(M - N) \right\}.$$

If  $Par^\infty(-)$  is obtained as the quotient of  $Par_1^\infty(-)$  for some particle functor of the first kind, then  $Par^\infty(M, N)$  is obtained as a pushout construction

$$\begin{array}{ccc} Par_1^\infty(M) & \longrightarrow & Par_1^\infty(M, N) \\ \downarrow & & \downarrow \\ Par^\infty(M) & \longrightarrow & Par^\infty(M, N). \end{array}$$

In words,  $\zeta \in Par^\infty(M, N)$  if  $\zeta \cap (M - N) \in Par^\infty(M - N)$  with the additional constraint that as points of  $\zeta$  tend to  $N$  they get identified with basepoint.

**Remark 6.1:** Notice that  $Par^\infty(M, N)$  has a canonical basepoint  $\zeta = \langle *, *, \dots \rangle$ . Observe as well that  $Par^\infty(M, N) \simeq Par^\infty(M/N, *)$  (compare 2.6).

**Lemma 6.2:** Let  $M \in \underline{\mathcal{C}}_n$ ,  $N \subset M$ . Then we have a quotient map  $\pi : Par^\infty(M) \longrightarrow Par^\infty(M, N)$ . If  $N$  has boundary  $\partial N$ , we get a restriction

$$6.3 \quad r : Par^\infty(M) \longrightarrow Par^\infty(N, \partial N).$$

PROOF: We simply need mention that  $r$  is a special case of  $\pi$  as applied to the quotient  $M \rightarrow M/(M - N)$  and one can check that  $Par^\infty(M, M - N) = Par^\infty(N, \partial N)$ . ■

**Remark 6.4:** We can give an explicit description of  $\pi$  as follows. Let  $\vec{\zeta} \in Par^\infty(M)$ . Then since  $Par^\infty(M) = Par^\infty(N) + Par^\infty(M - N)$ , we can write  $\vec{\zeta} = \vec{\zeta}_N + \vec{\zeta}_{M-N}$  where  $\vec{\zeta}_N \in Par^\infty(N)$  and  $\vec{\zeta}_{M-N} \in Par^\infty(M - N)$ . The correspondence

$$\vec{\zeta} \mapsto \vec{\zeta}_{M-N}$$

is not continuous. However when post-composed with the quotient map

$$Par^\infty(M - N) \longrightarrow Par^\infty(\overline{(M - N)}, \partial(\overline{(M - N)})) \cong Par^\infty(M, \overline{N}) = Par^\infty(M, N)$$

it becomes so, hence yielding 6.2 (here we use the fact that  $Par^\infty(M - N)$  is homeomorphic to  $Par^\infty(\overline{(M - N)})$ ). On the other hand, the correspondence  $\vec{\zeta} = \vec{\zeta}_N$  yields the *restriction* map  $r : Par^\infty(M) \longrightarrow Par^\infty(N, \partial N)$ .

**Remark 6.5:** There are different other restriction maps. For instance, let  $M_0 \subset N \subset M \in \underline{\mathcal{C}}_n$ , then we have maps as follows

$$Par^\infty(M, M_0) \longrightarrow Par^\infty(N, \partial N \cup M_0).$$

Notice also that given any morphism of pairs in  $\underline{\mathcal{C}}$ ,  $(M, N) \hookrightarrow (M', f(N))$  we get an induced morphism

$$Par^\infty(M, N) \longrightarrow Par^\infty(M', f(N)).$$

**§6.2 Behaviour with respect to cofibrations :** From now on we restrict attention to the subcategory  $\underline{\mathcal{C}}_n$ , and hence  $Par^\infty : \underline{\mathcal{C}}_n \rightarrow \underline{\mathcal{C}}$ . Associated to any pair  $(M, N) \in \underline{\mathcal{C}}_n$ , ( $N \subset M$  is of codimension 0), we have the cofibration sequence

$$N \hookrightarrow M \rightarrow M/N.$$

Using the covariance of  $Par^\infty$  with respect to inclusions and using the restriction map constructed early in this section, we can apply  $Par^\infty$  to the above sequence and get

$$Par^\infty(N) \longrightarrow Par^\infty(M) \longrightarrow Par^\infty(M, N).$$

More generally, we can start with the cofibration sequence

$$(N, N \cap M_0) \longrightarrow (M, M_0) \longrightarrow (M, N \cup M_0).$$

Then the following is a generalization of proposition 2.2, and [B] (p:178);

**Proposition 6.6:** *Consider the cofibration sequence  $(N, N \cap M_0) \longrightarrow (M, M_0) \longrightarrow (M, N \cup M_0)$  with  $N \subset M \in \underline{\mathcal{C}}_n$ ,  $M_0 \subset M$ . Suppose that  $M_0 \cap N \neq \emptyset$ , then*

$$Par^\infty(N, N \cap M_0) \longrightarrow Par^\infty(M, M_0) \xrightarrow{\pi} Par^\infty(M, N \cup M_0)$$

is a quasifibration.

PROOF: The submanifold  $N \subset M$  being proper and compact, it has non empty boundary  $\partial N$  which we can assume wlog to be connected.  $\partial N$  has a tubular neighborhood  $U_\partial \subset M$  when restricted to either  $N$  or  $M - N$  looks like a collar. Let  $U = N \cup U_\partial$ , then there is an isotopy retraction of  $r_t : U \longrightarrow N$  which leaves  $M - U$  and  $N$  invariant ([Ko], chap.3). Consider at this point the subspaces

$$X_{k_1, \dots, k_n} := \left\{ \vec{\eta} \in Par^\infty(M, N \cup M_0) \mid \vec{\eta} \cap (M - N \cup M_0) \in Par_{i_1, \dots, i_n}^\infty(M - N \cup M_0), i_j \leq k_j \right\}$$

(here  $n$  is determined by  $Par$ ) and consider the open sets in  $Par^\infty(M, N \cup M_0)$

$$U_{k_1, \dots, k_n} = \{(\zeta_1, \dots, \zeta_r) \in X_{\vec{k}} \mid (\zeta_1, \dots, \zeta_r) \text{ contains a non-empty subtuple in } Par^\infty(U)\}.$$

Write  $X_{\vec{k}} = X_{k_1, \dots, k_n}$  and similary  $U_{\vec{k}} = U_{k_1, \dots, k_n}$ . By construction, we have the following inclusions

$$X_{<\vec{k}} := \bigcup_i X_{k_1, \dots, k_{i-1}, \dots, k_n} \subset U_{\vec{k}} \subset X_{\vec{k}}$$

and it is easy to see that over  $X_{\vec{k}} - X_{<\vec{k}}$  the map  $\pi : Par^\infty(M, M_0) \rightarrow Par^\infty(M, M_0 \cup N)$  is a direct product  $(X_{\vec{k}} - X_{<\vec{k}}) \times Par^\infty(N, N \cap M_0)$ . This direct product structure is actually given as follows: To  $\vec{\zeta} \in X_{\vec{k}} - X_{<\vec{k}}$  and  $\vec{\eta} \in Par^\infty(N, N \cap M_0)$  we associate the multiconfiguration  $\vec{\zeta} + \vec{\eta} \in \pi^{-1}(X_{\vec{k}} - X_{<\vec{k}}) \subset Par^\infty(M, M_0)$ . The sum  $\vec{\zeta} + \vec{\eta}$  is well defined since the support of  $\vec{\zeta}$  is in  $M - N \cup M_0$  and the support of  $\vec{\eta}$  is in  $N$ . This yields the first half of criterion 2.3. It is left to analyze the attaching maps associated to this filtration.

One now sees that the isotopy retraction  $r_t$  moves  $\partial N$  away from itself;  $r_1(N) \subset N$ , and squeezes the collar  $U_\partial$  into  $N$ . This is done through a homotopy that is injective on  $M - U_\partial$  and so from earlier considerations it induces a retraction at the level of particle spaces  $r : U_{\vec{k}} \longrightarrow X_{<\vec{k}}$ . Let  $\vec{x} = (\eta_1, \dots, \eta_r) \in U_{\vec{k}}$  and write

$$(\eta_1 + \zeta_1, \dots, \eta_r + \zeta_r) \in \pi^{-1}(\vec{x}),$$

where the  $\zeta_i$  are in  $Par^\infty(N, N \cap M_0)$ . Since  $(\eta_1, \dots, \eta_r) \in U_{\vec{k}}$ , there exist a subtuple  $(D_1, \dots, D_r) \subset (\eta_1, \dots, \eta_r)$  such that  $D_i \in U - N$ . The retraction  $r$  moves the  $D_i$  inside  $N$  (so at time  $t = 1$ ,  $r_1(U) \subset N$ ) and hence at the level of preimages we have a lifting

$$\begin{aligned} Par^\infty(N, N \cap M_0) &\xrightarrow{\tilde{r}_1} Par^\infty(N, N \cap M_0) \\ (\zeta_1, \dots, \zeta_r) &\mapsto (r_1(\eta_1) + r_1(D_1), \dots, r_1(\eta_r) + r_1(D_r)) \end{aligned}$$



where  $\tilde{r}_1$  is the attaching map

$$6.7 \quad \tilde{r}_1 : Par^\infty(N, N \cap M_0) \xrightarrow{\cong} \pi^{-1}(\vec{x}) \longrightarrow \pi^{-1}(r_1(\vec{x})) \xrightarrow{\cong} Par^\infty(N, N \cap M_0)$$

Since  $M_0 \cap N \neq \emptyset$ , we use lemma 6.2 to deform  $(r_1(D_1), \dots, r_1(D_r))$  through a path in  $Par^\infty(N)$  to a tuple in  $M_0 \cap N$  and hence to basepoint in  $Par^\infty(N, M_0 \cap N)$ . This produces a homotopy inverse for  $\tilde{r}_1$  and the proposition follows by 2.2.  $\blacksquare$

**Remark 6.8:** When  $N \cap M_0$  is empty, then  $Par^\infty(N)$  does generally split into components. In this case, the attaching map  $\tilde{r}_1$  switches components and it has no homotopy inverse. We deal with this case in the next section.

**Remark 6.9:** The proposition is also not true if  $N$  is not of codimension 0 in  $M$ . For example, let  $Par^\infty = C^\infty$  be the functor of disjoint unordered points (see introduction). We show in 11.1 that  $C^\infty(D^n, \partial D^n) \simeq S^n$ . Suppose in this case that  $M = D^n$ ,  $N = \{(x_1, \dots, x_{n-1}, 0)\} \subset D^n$  is an  $n$ -th face and let  $M_0 = \partial D^n - N$ . If 6.6 were to apply in this case, then we get a quasifiber

$$C^\infty(D^{n-1}, \partial D^{n-1}) \longrightarrow C^\infty(D^n, M_0) \longrightarrow C^\infty(D^n, \partial D^n).$$

But since  $C^\infty(D^n, M_0)$  is contractible, we would have proved that  $S^{n-1}$  is weakly homotopy equivalent to  $\Omega S^n$  which is obviously false.

**§6.3 Connectivity properties :** At this point, we would like to analyze the connectivity of the spaces  $Par^\infty(M, *)$ . The following is a direct consequence of 6.6 above.

**Proposition 6.10:** *Let  $M$  be any  $n - 1$  connected finite CW complex ( $n > 1$ ). Then  $Par^\infty(M, *)$  is also  $n - 1$  connected.*

PROOF: The proof is a standard induction on cells of  $M$ . First since  $M$  is  $n - 1$  connected, it has a CW decomposition with cells starting in dimension  $n$  attaching to a basepoint  $*$ . Let  $M^{(i)}$  denote the  $i$ -th skeleton of  $M$  (here of course  $i \geq n$ ). The inclusion of  $M^{(i)}$  into the next skeleton gives a cofibration sequence

$$M^{(i)} \longrightarrow M^{(i+1)} \longrightarrow \bigvee S^{i+1}$$

which yields by 6.6 a quasifibration

$$6.11 \quad Par^\infty(M^{(i)}, *) \longrightarrow Par^\infty(M^{(i+1)}, *) \longrightarrow \prod Par^\infty(S^{i+1}, *), \quad i \geq n.$$

Suppose that  $Par^\infty(S^n, *)$  is  $n - 1$  connected, then  $Par^\infty(M^{(n)}, *) = \prod Par^\infty(S^n, *)$  is also  $n - 1$  connected and the long exact sequence in homotopy attached to 6.11 shows that  $Par^\infty(M^{(n+1)}, *)$  is  $n - 1$  connected as well. Proceeding inductively, we can establish the claim as soon as we show that  $Par^\infty(S^n, *)$  is  $n - 1$  connected. This is done in 9.2.  $\blacksquare$

## §7 “Partial” Group Completion

Start with the simplest particle space  $SP^\infty(M) = \coprod SP^n(M)$  and notice that

$$SP^\infty(M)^+ \simeq \mathbf{Z} \times SP^\infty(M, *)$$

where  $SP^\infty(M)^+$  means group completion with respect to the monoid structure. One way of constructing  $SP^\infty(M)^+$  is to consider the space of *infinite* configurations of the form  $\sum n_i x_i$  where the  $n_i$  are not necessarily positive. Choose a sequence  $\{z_i\}_{i=1}^\infty$  of points in  $M$ , and write  $\eta = \sum z_i$  for the corresponding infinite configuration. Then  $SP(M) = SP^\infty(M)^+$  is the set of infinite configurations  $\zeta$  such that  $\zeta - \eta$  is a finite but not necessarily positive configuration.

It is our desire to construct an analogue of  $SP(-)$  (which we denote by  $Par(-)$ ) for the more general particle functors  $Par^\infty(-)$ . The end result would be some sort of “group completion” with respect to the partial monoid structure of  $Par^\infty(M)$ . This is done in a very standard way.

As always, let  $M$  be compact (connected) and  $A \subset M$  a closed non-empty ANR (typically  $A = \partial M$  for example). We can “stabilize”  $Par^\infty(M)$  by “marching toward  $A$ ”. Let  $U$  be a tubular neighborhood of  $A$  which we assume to retract to  $A$  via a retraction  $r$  which is injective outside of  $U$ . Let  $U_{i \in \mathbb{Z}^+}$  be a nested sequence  $U_{i+1} \subset U_i \subset U$  The  $U_i - U_{i+1}$  being open,  $Par^\infty(U_i - U_{i+1}) \neq \emptyset$  according to lemma 5.1, and so we can choose  $\vec{\eta}_i \in Par^\infty(U_i - U_{i+1})$ . We choose  $\vec{\eta}_i$  to be “minimal” in the sense that no smaller subtuple of it lies in  $Par^\infty(U_i - U_{i+1})$ . Now notice that we have an inclusion given by summing with  $\vec{\zeta}_i$  in the partial monoid structure on  $Par^\infty(M - U_{i+1})$ ;

$$Par^\infty(M - U_i) \xrightarrow{+\vec{\zeta}_i} Par^\infty(M - U_{i+1})$$

We can now make the definition

DEFINITION 7.1: For  $M$ ,  $A$  and  $U$  as above, we define

$$Par(M) = \lim_{\vec{\zeta}_i} \left( Par^\infty(M - U_i) \xrightarrow{+\vec{\zeta}_i} Par^\infty(M - U_{i+1}) \right).$$

**Remark 7.2:** It should be clear that  $Par(M)$  doesn’t depend (up to homeomorphism) on the choice (up to isotopy) of the stabilizing sequence  $\vec{\eta}_i$  or of the nested sequence  $\{U_i\}$ . It is equally clear (for the same reasons given earlier for the case  $Par^\infty = SP^\infty$ ) that components of  $Par(M)$  are homeomorphic.

**Remark 7.3:** We can define  $Par(M, N)$  for pairs  $(M, N)$  by taking suitable direct limits over  $Par^\infty(M - U_i, N - U_i \cap N)$ . When  $A = \partial M$  for example (or a subset of it),  $N \cap \partial M \neq \emptyset$ , then we can stabilize with respect to a sequence of multiconfigurations  $\{\eta_i\}$  converging to a point  $p \in N \cap \partial M$ . By a homotopy (again injective in the complement of  $U$ ) we can retract points of  $\eta_j$  to  $p$  and this shows (in this case) that

$$Par(M, N) \simeq Par^\infty(M, N).$$

**Remark 7.4:** One may observe that if  $p \in \partial M \neq \emptyset$ , then we can stabilize with respect to a tuple  $\vec{\eta}_i$  converging to  $p$  (that is the sequence of points making up each  $\eta_i$  converges to  $p$ ). In this case one can show that

$$Par_c(M) \simeq Par^\infty(M, p)$$

EXAMPLE 7.5: We consider the case  $C^\infty(M) \subset SP^\infty(M)$ . We assume  $M$  has boundary  $\partial M \neq \emptyset$  and stabilize as above with respect to a collar  $U$  of  $M$  by choosing a sequence of *distinct* points  $z_i$  marching towards  $\partial M$ . Addition of points  $z_i$  yields maps and commutative diagrams

$$\begin{array}{ccc} C_i(M) & \xrightarrow{+z_i} & C_{i+1}(M) \\ \downarrow \subset & & \downarrow \subset \\ SP^i(M) & \xrightarrow{+} & SP^{i+1}(M). \end{array}$$

which in the limit yield a map  $C(M) \longrightarrow SP(M)$ . Now it isn't hard to see that  $C(M)$  breaks into  $\mathbf{Z}$  components which we write  $C^\infty(M, p)$  for some  $p \in \partial M$  (see 7.4). We then get a map

$$\alpha : C^\infty(M, p) \longrightarrow SP^\infty(M, p)$$

It is interesting to notice that in the case  $M = \mathbf{R}^n$  for example, the left hand side is equivalent in homology to a component of  $\Omega^n S^n$  (cf. §12) and hence  $\alpha_n$  is homologous to the map

$$\Omega_0^n S^n \longrightarrow SP^\infty(\mathbf{R}^n, *) \simeq *$$

obtained by looping  $n$  times the natural inclusion  $S^n \longrightarrow SP^\infty(S^n, *)$ .

DEFINITION: A morphism  $\iota : Par(N) \longrightarrow Par(M)$  will mean an inclusion such that  $\iota(\eta_N) = \eta_M$ . The following is a generalization of 6.6

**Theorem 7.6:** *Given a cofibration sequence  $N \longrightarrow M \longrightarrow (M, N)$ ,  $N, M \in \mathcal{C}_n$ , and assume  $\partial N \neq \emptyset$ . Then*

$$Par(N) \xrightarrow{i} Par(M) \xrightarrow{\pi} Par^\infty(M, N)$$

*is a homology fibration.*

PROOF: First we convince ourselves that the preimage under  $\pi$  is homeomorphic to  $Par(N)$  and so  $i$  is constructed up to homeomorphism. The proof now amounts to showing that the attaching map  $\tilde{r}_1 : Par(N) \rightarrow Par(N)$  is a homology equivalence. Recall (§6) that  $\tilde{r}_1$  is obtained by moving particles of  $N$  away from a collar  $U$  of  $\partial N$  and then adding a given element  $\vec{v} \in Par^\infty(r_1(U - N))$ .

Let  $\vec{\eta}_i$  be a stabilizing sequence used in constructing  $Par(N)$  with respect to some tubular neighborhood of  $\partial N$ . Then up to isotopy,  $\vec{\eta}$  can be written as a finite sum over some  $\vec{\eta}_j$ 's. To see this, we observe that via an isotopy (if necessary) we can bring points of  $\vec{\eta}_1$  (for example) to points of  $\vec{\eta}$  and hence  $\vec{\eta} - \vec{\eta}_1$  will be positive (by minimality of  $\eta_1$ ) and belongs to  $Par^\infty(N)$ . Reiterating this argument shows that  $\vec{\eta} = \sum \vec{\eta}_j$  (a finite sum). By construction of  $Par(N)$  as a direct limit over addition of the  $\vec{\eta}_j$ 's,  $+\vec{v}$  necessarily induces a homology isomorphism and the claim follows. ■

**Remark 7.7:** We emphasize again that  $N$  needs to be of codimension 0 in  $M$  (see remark 6.9). Here's (another) example where 7.6 doesn't hold if  $N$  not of zero codimension. Consider the particle space  $Div^2(M) \subset SP^\infty(M) \times SP^\infty(M)$  consisting of pairs of configurations on  $M$  with no points in common and let  $N = * \hookrightarrow M$ . Then  $Div^2(N) = Div^2(*) = \emptyset$  and there isn't much sense to the

sequence  $Div^2(*) \rightarrow Div^2(M) \rightarrow Div^2(M, *)$ . However if we replace  $*$  by the open disc  $D^n$ , then  $Div^2(D^n)$  has now reasonable properties; in fact it is given by

$$\Omega^n(K(\mathbf{Z}, n) \vee K(\mathbf{Z}, n)) \quad (cf. \S 12)$$

and  $Div^2(D^n) \longrightarrow Div^2(M) \rightarrow Div^2(M, *)$  is a quasifibration.

## §8 Scanning Smooth Manifolds

The term “scanning” is borrowed from Segal ([S1]). We assume as usual that  $M$  is smooth and connected  $n$ -dimensional manifold.

**§8.1. Scanning parallelizable manifolds:** The scanning process is best pictured when  $M$  is parallelizable (i.e.  $M$  has trivial tangent bundle). Examples of such manifolds are Lie groups or any oriented three dimensional manifold. Without loss of generality, we restrict attention below to  $Par^\infty(M) = SP^\infty(M)$  (the more general situation is treated in the exact same way).

**Definition 8.1:** Let  $M^n$  be as above and let  $N$  be a closed subset of  $N$ . The *pair*  $(M, N)$  is said to be *parallelizable* if  $M - N$  is parallelizable.  $M$  is *stably parallelizable* if  $(M, *)$  is parallelizable for instance. Riemann surfaces are examples of stably parallelizable surfaces, as well as compact, oriented, spin four manifolds.

Put a metric on  $M$  and consider the unit disc bundle  $\tau M$  lying over  $M$ . Let's assume for now that  $\partial M = \emptyset$ .) Via the exponential map we can identify a neighborhood of every point  $x \in M$  with the fiber at  $x$ . Denote such a neighborhood by  $D(x) \subset M$ . When  $M$  is parallelizable, the fibers over  $\tau M$  are canonically identified with a disc  $D^n$  and hence one can identify canonically the pairs  $(\bar{D}(x), \partial \bar{D}(x))$  for every  $x \in M$  with  $(D^n, \partial D^n) = (S^n, \infty)$  (where the north pole  $\infty$  is chosen to be the basepoint in  $S^n$ .)

Given a configuration  $\zeta \in SP^d(M)$  and an  $x \in M$ , then  $\zeta \cap D(x)$  is a configuration on  $D(x)$  and its image under the restriction map

$$SP^\infty(D(x)) \longrightarrow SP^\infty(\bar{D}(x), \partial \bar{D}(x)) = SP^\infty(S^n, \infty)$$

is denoted by  $\zeta_x$ . Notice that the correspondence  $\zeta \mapsto \zeta_x$  is now continuous (while the correspondence  $\zeta \mapsto \zeta \cap D^n(x)$  was not to begin with). Starting with  $\zeta \in SP^d(M)$ , we hence get a map

$$S_d : SP^d(M) \longrightarrow \text{Map}_d(M, SP^\infty(S^n, *)), \quad \zeta \mapsto f_\zeta : f_\zeta(x) = \zeta_x.$$

The scanning map  $S$  is now given as  $\sqcup S_r$ .

**Scanning manifolds with boundary:** We're in the case  $\partial M \neq \emptyset$ . We can still scan the interior  $M - \partial M$  and alter the topology as points tend to  $\partial M$ .

Consider the open interior  $M^{int} = M - \partial M$  and let  $SP_\epsilon^r(M^{int})$  be the subspace of  $SP^r(M)$  consisting of configurations of points that are at least  $2\epsilon$  away from the boundary  $\partial(M)$ . Choose

$\zeta \in SP_\epsilon^r(M^{int})$ . Then by scanning the interior using discs of radius  $\epsilon$ , it is clear that  $S_\zeta$  maps  $x$  to basepoint for  $x$  sufficiently near the boundary. This gives rise to a map

$$SP_\epsilon^r(M^{int}) \longrightarrow \text{Map}_r(M/\partial M, SP^\infty(S^n, *)).$$

As  $\epsilon \rightarrow 0$ , one obtains in the limit a map

$$S_r : SP^r(M) \longrightarrow \text{Map}_r(M/\partial M, SP^\infty(S^n, *)) \simeq \text{Map}_r(M/\partial M, K(\mathbf{Z}, n)).$$

In exactly the same way, one obtains for each parallelizable pair  $(M, N)$  a map

$$S : SP^\infty(M - N) \longrightarrow \text{Map}(M, N \cup \partial M, SP^\infty(S^n, *))$$

where the right hand side consists of all based maps sending  $N$  into basepoint  $* \in SP^\infty(S^n, *)$ .

**Scanning smooth manifolds:** The general case of  $M$  not necessarily parallelizable and  $Par^\infty$  any particle functor is treated similarly. The starting point is again the unit disc bundle  $\tau M$ . Compactifying each fiber yields a bundle

$$S^n \longrightarrow \widehat{\tau M} \longrightarrow M$$

to which we associate the bundle of configurations

$$8.2 \quad Par^\infty(S^n, *) \longrightarrow E_{Par^\infty} \longrightarrow M.$$

by applying  $Par^\infty$  fiberwise. Note that  $\widehat{\tau M}$  has a “zero” section  $\nu$  sending each  $x \in M$  to the compactifying point in the fiber. We label this point by  $*$ . Clearly, such a section extends to a zero section of  $Par^\infty(S^n, *) \longrightarrow E \longrightarrow M$  also denoted by  $\nu$ . We denote by  $\text{Sec}(M, A, Par^\infty(S^n, *))$  the space of sections restricting to  $\nu$  on  $A \subset M$ .

The exponential map again provides a cover of  $M$  by neighborhoods  $\bigcup_{x \in M} D^n(x)$  with respect to which we can scan. Cutting a neighborhood  $\bar{D}^n \subset M$  yields a cofibration  $M - \bar{D}^n \hookrightarrow M \rightarrow (\bar{D}^n, \partial \bar{D}^n)$  and hence we get “restriction” maps

$$\pi_x : Par^\infty(M) \longrightarrow Par^\infty(\bar{D}^n(x), \partial \bar{D}^n(x)), \quad \forall x \in M$$

Starting with an element in  $Par^\infty M$ , one can restrict via  $\pi_x$  to neighborhoods as in 8.1. The elements of  $\text{Map}(D^n(x), Par^\infty(S^n, *))$  are now local sections of 8.2 and one gets the correspondence

**Lemma 8.3:** Let  $M \in \underline{\mathcal{C}}_n$  and  $N \subset M$  a closed ANR. Then scanning yields a map

$$Par^\infty(M - N) \longrightarrow \text{Sec}(M, N \cup \partial M, Par^\infty(S^n, *)).$$

**§8.2 The Electric Field Map:** The scanning map is closely associated to another map of Segal ([S1]). Given  $k$  points in  $\mathbf{R}^n$ , we can electrically charge them and hence we can associate to them an electric field  $E$  which is a function on  $\mathbf{R}^n$  taking values in  $\mathbf{R} \cup \infty$  (where  $\infty$  is reached at the charge points). Since the electric field intensity decays away from the charges, we can extend the previous function to  $\mathbf{R}^n \cup \infty$  by taking  $\infty$  to the basepoint  $0 \in S^n$ . So to  $k$ -points  $\{q_1, \dots, q_k\}$  in  $\mathbf{R}^n$ , we

associated the map  $E_{q_1, \dots, q_k} : S^n \rightarrow S^n$  which is based and of degree  $k$  (since  $E^{-1}(\infty) = \{q_1, \dots, q_k\}$ ). This then shows that  $E_{q_1, \dots, q_r} \in \Omega_k^n(S^n)$  as desired. It is easy to see that  $S$  corresponds to  $E_k$ ; that is

**Lemma 8.4:** *The maps  $S, E_k : C_k(\mathbf{R}^n) \rightarrow \Omega_k^n(S^n)$  are homotopic.*

PROOF: Let  $C_\epsilon(\mathbf{R}^n, k)$  be the subset of divisors  $D \in C(\mathbf{R}^n, k)$  such that  $D = p_1 + \dots + p_k$ ,  $p_i \neq p_j$  and  $|p_i - p_j| > \epsilon$ . We consider the electric field map  $E_D$  associated to  $D$ . Let  $B_\epsilon(x)$  be the ball of radius  $\epsilon$  around the point  $x \in \mathbf{R}^n$ . Then by shrinking the vector field one can confine it inside the balls  $B_\epsilon(p_i)$  so that it is zero outside of these balls. If we choose the electric fields to die out linearly, then the map  $E_D$  is seen to correspond to scanning the configuration  $D$  with a ball of radius  $\epsilon$ . Now since  $C(\mathbf{R}^n, k) = \bigcup_{\epsilon > 0} C_\epsilon$  the lemma follows. ■

## §9 Proof of Main Theorems 1.1 and 1.3

NOTATION: Recall that  $Par^\infty(S^n, *)$  has a “preferred” identity  $*$ . For each pair of spaces  $(M, N)$ , we will write  $\text{Map}(M, N, Par^\infty(S^n, *))$  for the space of continuous maps from  $M$  into  $Par^\infty(S^n, *)$  which send  $N$  to  $*$ .

**§9.1 The Homology Equivalence 1.1:** A good starting point is a quick review of the handle decomposition of a manifold (à la Thom-Smale-Milnor): Let  $M$  be a given smooth, compact manifold of dimension  $n$ . Then there exists a sequence of manifolds with boundary

$$M_0 = D^n \subset M_1 \subset \dots \subset M_{n-1} \subset M$$

such that  $M_i$  for  $0 < i < n$  is obtained from  $M_{i-1}$  by attaching a number of  $i$ -handles to its boundary. The manifold  $M$  is now obtained from  $M_{n-1}$  by attaching an  $n$ -dimensional cell. If  $M$  has boundary, then one sets  $M_{n-1} = M$ .

A handle  $H^i$  of index  $i$  is a copy of  $D^n = D^i \times D^{n-i}$  which is attached to  $\partial M_{i-1}$  via an embedding  $h : S^{i-1} \times D^{n-i} \hookrightarrow \partial M_{i-1}$ . The resulting space is a smooth manifold. The pair  $(D^i, S^{i-1})$  is called the *core* of the handle, while  $S^{i-1} \times D^{n-1}$  is referred to as the *attaching region*.

**Proposition 9.1:** *Let  $D^n$  be the closed unit disc. For  $0 < k \leq n$  we have homotopy equivalences*

$$Par^\infty(D^n, S^{k-1} \times D^{n-k}) \simeq \Omega^{n-k} Par^\infty(S^n, *),$$

*while for  $k = 0$  we have a homology equivalence  $H_*(Par(D^n); \mathbf{Z}) \simeq H_*(\Omega^n(Par^\infty(S^n, *)); \mathbf{Z})$ .*

PROOF: Let  $H^k$  be a handle  $D^n = D^k \times D^{n-k}$  of index  $k$  attached along  $(\partial D^k) \times D^{n-k}$ . We write  $A^k = S^{k-1} \times D^{n-k}$  and the goal is then to show that  $Par(H^k, A^k) \simeq \Omega^{n-k} Par^\infty(S^n, *)$  for  $0 \leq k \leq n$  (note that  $A^0 = \emptyset$ ).

The proof uses the cofibration sequence described in [B]. Let  $I_k \subset D^n = [0, 1]^n$  denote the subset of  $(y^1, \dots, y^n)$  such that  $y^i = 0$  or  $y^i = 1$  for some  $i = 1, \dots, k-1$ , or  $y^k = 1$  (that is  $I_k$  consist of all the boundary faces of  $D^k \subset D^n = D^k \times D^{n-k}$  safe the face  $y^k = 0$ ). Now let

$H_k = [0, 1]^{k-1} \times [0, \frac{1}{2}] \times [0, 1]^{n-k}$ . Then there is a cofibration sequence

$$9.2 \quad (H_k, H_k \cap I_k) \longrightarrow (D^n, I_k) \longrightarrow (D^n, H_k \cup I_k).$$

The pair  $(H_k, H_k \cap I_k)$  can be identified with  $(D^n, S^{k-2} \times D^{n-k+1})$  hence representing a  $k-1$ -handle  $(H^{k-1}, A^{k-1})$ , while the pair  $(D^n, H_k \cup I_k) = (D^n, S^{k-1} \times D^{n-k})$  represents a handle  $(H^k, A^k)$  of index  $k$ . Applying  $Par^\infty$  to 9.2 yields the quasifibration (theorem 6.6)

$$Par^\infty(H^{k-1}, A^{k-1}) \longrightarrow Par^\infty(D^n, I_k) \xrightarrow{\pi} Par^\infty(H^k, A^k).$$

The proof now proceeds by downward induction on  $k$ . Observe first that 9.1 is obviously true when  $k = n$ . Suppose it is true for some  $n \geq k > 0$ . Then one can consider the following diagram of quasifibrations

$$9.3 \quad \begin{array}{ccc} Par^\infty(H^{k-1}, A^{k-1}) & \longrightarrow & \Omega^{n-k+1} Par^\infty(S^n, *) \\ \downarrow & & \downarrow \\ Par^\infty(D^n, I_k) & \longrightarrow & PS \\ \downarrow \pi & & \downarrow \\ Par^\infty(H^k, A^k) & \longrightarrow & \Omega^{n-k}(Par^\infty(S^n, *)) \end{array}$$

By induction the bottom map is a homotopy equivalence while the middle map is also a homotopy equivalence since  $Par^\infty(D^n, I_k)$  is contractible whenever  $k \geq 1$  (this follows from the fact that when  $k \geq 1$ ,  $I_k \neq \emptyset$  and there is a retraction of  $D^n$  onto  $I_k$  which is injective on the complement of a tubular neighborhood of  $I_k$ ). It follows that the top inclusion must be a homotopy equivalence whenever  $k \geq 1$  (this establishes the first claim in 9.1).

Remains to treat the case  $k = 1$ . Since  $A^0 = \emptyset$  the left hand side in 9.3 is not a quasifibration anymore and we have to pass to the  $Par$  functor. We can then consider the diagram of fibrations

$$\begin{array}{ccc} F & \longrightarrow & \Omega^n Par^\infty(S^n, *) \\ \downarrow & & \downarrow \\ Par(D^n, I_1) & \longrightarrow & PS \\ \downarrow \pi & & \downarrow \\ Par^\infty(H^1, A^1) & \xrightarrow{\simeq} & \Omega^{n-1}(Par^\infty(S^n, *)) \end{array}$$

where  $F \simeq \Omega^n Par^\infty(S^n, *)$  is the homotopy fiber for the l.h.s and  $Par(D^n, I_1) \cong Par^\infty(D^n, I_1)$  (see 7.3) is contractible. Since the l.h.s is a homology fibration (theorem 7.6), the inclusion of the preimage  $Par(D^n, A^0) = Par(D^n)$  into  $F$  is a homology equivalence and the proof is complete. ■

**Corollary 9.4:**  $Par^\infty(S^n, *)$  is  $n-1$  connected.

PROOF: We have that  $\pi_k(Par^\infty(S^n, *)) = \pi_0(\Omega^k Par^\infty(S^n, *))$  and the latter is trivial whenever  $k < n$  since  $\Omega^k Par^\infty(S^n, *)$  is identified with  $Par^\infty(D^{n-k} \times D^k, S^{n-k-1} \times D^k)$  and the latter is connected by lemma 5.6. ■

EXAMPLE: When  $Par^\infty = C^\infty$  (again), we show in 11.1 that  $C^\infty(S^n, *) \simeq S^n$  (this is an early observation of Segal and McDuff). Naturally  $S^n$  is  $n-1$  connected and 9.1 shows that for  $0 \leq k < n$

$$C^\infty(D^n, D^k \times S^{n-k-1}) \simeq \Omega^k S^n.$$

**Theorem 9.5:** Let  $M \in \mathcal{C}_n$  and  $N$  a closed ANR in  $M$ . Suppose either  $N$  or  $\partial M$  non-empty. Then scanning induces a homology equivalence

$$S_* : H_*(Par(M - N); \mathbf{Z}) \xrightarrow{\cong} H_*(Sec(M, N \cup \partial M, Par^\infty(S^n, *)); \mathbf{Z}).$$

PROOF: Since  $Par$  is an isotopy functor, then  $Par(M - N) = Par(M - T(N))$  where  $T(N)$  is a tubular neighborhood of  $N$  and so wlog we can assume that  $N$  is of codimension 0. We consider the case  $N \neq \emptyset$  and  $\partial M = \emptyset$  (the other cases are treated similarly). Since  $Par(M - N) = Par(M - int(N))$ , we can assume that  $M - N$  is compact and has boundary  $\partial N$ . So  $M - N$  has a handle decomposition

$$M_0 = D^n \subset M_1 \subset \cdots \subset M_{n-1} = M - N$$

where the handles we attach have index at most  $n-1$  and all the  $M_i$  have boundary  $\partial M_i \neq \emptyset$ . The proof now proceeds by induction on  $i$ . Since the number of handles we attach at each stage (finitely many) is immaterial for the arguments below we might as well assume we're only attaching one handle at a time. That is

$$M_i = M_{i-1} \cup H^i, \text{ and } \partial M_{i-1} \cap H^i = S^{i-1} \times D^{n-i}.$$

Consider the following two cofibrations;

$$9.6 \quad M_{i-1} \longrightarrow M_i \longrightarrow (D^i \times D^{n-i}, S^{i-1} \times D^{n-i}), i < n,$$

and the one induced from the handle attachment

$$9.7 \quad (H^i, H^i \cap \partial M_i) \longrightarrow (M_i, \partial M_i) \longrightarrow (M_i, H^i \cup \partial M_i) = (M_{i-1}, \partial M_{i-1}).$$

We now apply the functor  $Sec$  to 9.7 and get the *fibration*

$$9.8 \quad Sec(M_{i-1}, \partial M_{i-1}, Par^\infty(S^n, *)) \rightarrow Sec(M_i, \partial M_i, Par^\infty(S^n, *)) \rightarrow Sec(H^i, H^i \cap \partial M_i, Par^\infty(S^n, *)).$$

Since  $E_{Par^\infty(H^i)}$  over  $H^i$  is trivial, we can replace  $Sec(H^i, H^i \cap \partial M_i, Par^\infty(S^n, *))$  by an iterated loop space as follows

$$\begin{aligned} \text{Map}(H^i, H^i \cap \partial M_i; Par^\infty(S^n, *)) &= \text{Map}(D^i \times D^{n-i}, \partial D^{n-i}; Par^\infty(S^n, *)) \\ &= \text{Map}^*(S^{n-i} \times D^i; Par^\infty(S^n, *)) \\ &= \Omega^{n-i} Par^\infty(S^n, *). \end{aligned}$$

On the other hand one can apply the functor  $Par$  to 9.6 and obtain a quasifibration which maps via scanning into 9.8 as follows

$$9.9 \quad \begin{array}{ccc} Par(M_{i-1}) & \xrightarrow{S} & Sec(M_{i-1}, \partial M_{i-1}; Par^\infty(S^n, *)) \\ \downarrow & & \downarrow \\ Par(M_i) & \xrightarrow{S} & Sec(M_i, \partial M_i; Par^\infty(S^n, *)) \\ \downarrow & & \downarrow \\ Par^\infty(D^i \times D^{n-i}, S^{i-1} \times D^{n-i}) & \xrightarrow{\simeq} & \Omega^{n-i} Par^\infty(S^n, *). \end{array}$$



The bottom map is a homotopy equivalence whenever  $1 \leq i \leq n$  by 9.1. When  $i = 1$ ,  $M_{i-1} = D^n$  and the top map is a homology equivalence (here  $\text{Sec}(M_0, \partial M_0; \text{Par}^\infty(S^n, *))$  is again identified with  $\Omega^n \text{Par}^\infty(S^n, *)$ .) By a standard spectral sequence argument, it follows that the middle map  $\text{Par}(M_1) \longrightarrow \text{Sec}(M_1, \partial M_1; \text{Par}^\infty(S^n, *))$  is a homology equivalence and the argument proceeds by induction. ■

**Remark 9.10:** The theorem above is not true if both  $N$  and  $\partial M$  are empty. In that case ( $M$  is closed)  $\text{Par}(M)$  is not even defined and it doesn't even hold true that the components of  $\text{Map}(M, \text{Par}^\infty(S^n, *))$  are homotopy equivalent. For example, we show in [K2] that components of  $\text{Map}(M_g, \mathbf{P}^n)$  (which is a special case of 11.7) do differ in homotopy type.

**Theorem 9.11:** *Let  $N, M$  be as in 9.4 and suppose  $\pi_1(\text{Par}(\mathbf{R}^n))$  is abelian, then scanning is a homotopy equivalence*

$$\text{Par}(M - N) \xrightarrow{\simeq} \text{Sec}(M, N \cup \partial M, \text{Par}^\infty(S^n, *)).$$

PROOF: Consider 9.9 again and the case  $i = 1$ . When  $\pi_1(\text{Par}(\mathbf{R}^n))$  is abelian, the l.h.s in 9.9 becomes a quasifibration (this is explained in 9.12 below) and hence the top map is a weak homotopy equivalence. Since the spaces involved have the homotopy type of CW complexes we get a homotopy equivalence and hence an equivalence in the middle. The rest of the proof is obtained by induction knowing that the bottom map is always a homotopy equivalence when  $1 \leq i \leq n - 1$ . ■

**§9.2 Good Functors:** The functor  $\text{Par}$  is *good* if it turns cofibrations  $N \rightarrow M \rightarrow M/N$ ,  $N, M \in \mathcal{C}_n$  into quasifibrations. A straightforward examination of the proof of 9.5 shows that scanning induces a weak homotopy equivalence

$$\text{Par}(M - N) \xrightarrow{\simeq} \text{Sec}(M, N \cup \partial M, \text{Par}^\infty(S^n, *))$$

whenever  $\text{Par}$  is good and  $N$  and  $M$  are as in 9.5 (note that the space of sections has the homotopy type of a CW complex; cf. [BS], lemma 6.5). The condition needed in theorem 9.11 is of course slightly weaker.

**Lemma 9.12:** *The functor  $\text{Par}$  is good if it abelianizes fundamental groups; that is if  $\pi_1(\text{Par}(M))$  is abelian for any  $M \in \mathcal{C}_n$ .*

PROOF: We need show that  $\text{Par}$  applied to cofibrations yields quasifibrations. This boils down to showing that the attaching maps given by addition of particles (see 6.7) are homotopy equivalences.

These attaching maps which take the form  $\text{Par}(M) \xrightarrow{+\zeta} \text{Par}(M)$  are homology equivalences for any twisted coefficients (by construction of  $\text{Par}(M)$  as a direct limit over these additions). When  $\pi_1(M)$  is abelian, the map  $+\zeta$  induces an isomorphism of fundamental groups as well. This implies that the attaching maps must be a homotopy equivalences and the lemma follows. ■

**Example 9.13:** Since  $\pi_1(SP^\infty(X)) = H_1(X)$  (for any space  $X$ ) we automatically have that  $SP$  is a good functor. More is true in this case for one can show that  $\pi_1(SP^n(X))$  is already abelian

when  $n \geq 2$ . To see this, let  $\alpha \in \pi_1(SP^n(X))$ , and  $q : X^n \rightarrow SP^n(X)$  for the quotient map. The loop  $\alpha$  (representing a class in  $\pi_1$ ) can be homotoped away from the branched points for  $n > 1$  (with basepoint  $*$  fixed) and hence it can be lifted to  $X^n$ . Since  $q^{-1}(*) = *$ , it follows that  $\alpha$  lifts to a loop in  $X^n$ . The rest of the claim follows from this observation.

**Example 9.14:** The functor  $C$  is not “good” and so the homology equivalence of 9.5 for the case  $Par = C$  cannot be upgraded in general to a homotopy equivalence. A standard example is already provided by the closed unit disc  $D^n$ . Consider  $C(\mathbf{R}^n) \cong C(D^n)$  and hence from theorem 9.5 we infer that

$$H_*(C(\mathbf{R}^n); \mathbf{Z}) \cong H_*(\Omega^n S^n; \mathbf{Z}).$$

At the level of components  $C(D^n) = \mathbf{Z} \times C^\infty(D^n, *)$  where  $C^\infty(D^n, *)$  is as described in example 7.4. It is known that  $\pi_1(C^\infty(D^n, *)) \cong \Sigma_\infty$  for  $n > 2$  (and is the braid group for  $n = 2$ ). Since  $\pi_1(\Omega^n S^n) \cong \mathbf{Z}_2$ , theorem 9.5 in this case couldn't possibly be upgraded to a homotopy equivalence.

**§9.3 Identifying Components:** The equivalence in 9.5 gives a homology equivalence at the level of components. We identify these components for both  $Par(M - N)$  and the space of sections. For the sake of simplicity we confine ourselves to the case  $M - N$  parallelizable.

**Lemma 9.14:** *Let  $X$  be a connected topological space,  $M, N$  as above,  $N \neq \emptyset$ . Then all components of  $Map(M, N, Par^\infty(S^n, *))$  are homotopy equivalent.*

PROOF: Let  $* \in N \subset M$  and identify  $*$  with  $N$  in  $M/N$ . Pick a map  $f \in Map$  and observe that for a small disc  $D \in M - N$ ,  $f(\partial D)$  is null homotopic in  $Par^\infty(S^n, *)$  (since the latter is  $n - 1$  connected). So  $f|_{\partial D}$  extends out to a map of a sphere  $S^n$  and if we denote by  $\#$  the connected sum, we have a map

$$Map^*(M/N, Par^\infty(S^n, *)) \longrightarrow Map^*((M/N) \# S^n, Par^\infty(S^n, *))$$

which takes one component to the next (here of course  $(M/N) \# S^n \simeq M/N$ ). This map is a homotopy equivalence for it can be reverted by attaching another sphere with reverse orientation.

■

**Proposition 9.15:** *Let  $\overline{M - N}$  be the closure of  $M - N$  and let  $p \in \partial(\overline{M - N}) \neq \emptyset$ . Then*

$$S : Par^\infty(\overline{M - N}, p) \xrightarrow{S} Map_0(M, N, Par^\infty(S^n, *))$$

*is a homology equivalence. Here  $Map_0$  stands for the component of null-homotopic maps.*

PROOF: Recall (§7) that the construction of  $Par$  involved a choice of multiconfigurations  $\vec{\zeta}_i \in U_i \subset U$  where  $U$  is an open collar around  $N$  and the  $U_i$  form a nested sequence contracting to  $U$ . The stabilization maps  $+\vec{\zeta}_i$  are easily seen to commute with scanning and hence we get a commuting diagram ( $Map_c$  and  $Map_{c'}$  are some components)

$$\begin{array}{ccc}
Par^\infty(M - U_i) & \xrightarrow{S_i} & Map_c(M/U_i, Par^\infty(S^n, *)) \\
\downarrow +\vec{\zeta}_i & & \downarrow \\
Par^\infty(M - U_{i+1}) & \xrightarrow{S_{i+1}} & Map_{c'}(M/U_{i+1}, Par^\infty(S^n, *)).
\end{array}
\tag{9.16}$$

Observe that  $Par^\infty(M - U_i) \cong Par^\infty(M - N)$  for all  $i$  and up to homotopy we have  $Map_c = Map_0$ . We then get the homotopy commutative diagram

$$\begin{array}{ccc} Par^\infty(M - N) & \xrightarrow{S} & Map_0(M, N, Par^\infty(S^n, *)) \\ \downarrow +\vec{\zeta}_i & & \downarrow \\ Par^\infty(M - N) & \xrightarrow{S} & Map_0(M, N, Par^\infty(S^n, *)). \end{array}$$

which yields in the limit a map of components

$$Par^\infty(\overline{M - N}, p) \xrightarrow{S} Map_0(M, N, Par^\infty(S^n, *))$$

and this must be a homology equivalence. ■

**Remark 9.17:** Since  $Par^\infty(S^n, *)$  is  $n - 1$  connected (proposition 6.10) it follows that

$$\pi_0 Map(M, N, Par^\infty(S^n, *)) = [M/N, Par^\infty(S^n, *)]_*$$

and hence that the connected components of the corresponding mapping space (and consequently of  $Par(M - N)$ ) are indexed by maps of  $H_n(M, N; \mathbf{Z})$  into  $H_n(Par^\infty(S^n, *))$ .

## §10 Duality on Manifolds

As mentioned in the introduction, theorem 9.5 admits a strengthening when  $Par^\infty = SP^\infty$ . This last functor is a homotopy functor on the one hand, and on the other it takes values in abelian monoids. We start with some standard results.

First we point out that since  $K(\mathbf{Z}, n)$  has the homotopy type of an abelian group, then so does the space of maps  $Map(X, K(\mathbf{Z}, n))$  and for connected  $X$ , all components of  $Map(X, K(\mathbf{Z}, n))$  are homotopy equivalent. A classical result (attributed to Moore) asserts that any abelian topological group is a product of Eilenberg-MacLane spaces. It remains to determine what these EM spaces are for the case of  $Map(X, K(\mathbf{Z}, n))$  and this is exactly the content of the following theorem of Thom (cf. [NS])

**Theorem 10.1:** (Thom) *Let  $X$  be connected,  $\pi$  an abelian group and  $n > 0$ . Then*

$$Map(X, K(\pi, n)) \simeq \prod_{0 \leq i \leq n} K(H^{n-i}(X, \pi), i)$$

*and each component is given by the sub-product  $1 \leq i \leq n$  in the expression above.*

Let  $X = K(\mathbf{Z}, n)$  and consider the subspace  $Aut(K(\mathbf{Z}, n)) \subset Map(K(\mathbf{Z}, n), K(\mathbf{Z}, n))$  of self-homotopy equivalences of  $K(\mathbf{Z}, n)$ . This is an abelian subgroup and hence is also a product of EM spaces. Our next proposition is an earlier result of May [Ma] which we state and prove in “non-simplicial” terms.

**Proposition 10.2:** *We have the following commutative diagram of inclusions and equivalences*

$$\begin{array}{ccc} K(\mathbf{Z}, n) \times \text{Aut}(\mathbf{Z}) & \hookrightarrow & K(\mathbf{Z}, n) \times \text{Hom}(\mathbf{Z}, \mathbf{Z}) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Aut}(K(\mathbf{Z}, n)) & \hookrightarrow & \text{Map}(K(\mathbf{Z}, n), K(\mathbf{Z}, n)) \end{array}$$

PROOF: To simplify notation we write  $K_n := K(\mathbf{Z}, n)$ . From 10.1 and since  $H^{n-i}(K_n; \mathbf{Z}) = \mathbf{Z}$  when  $i = 0$  and zero otherwise, we get

$$\text{Map}(K_n, K_n) \simeq K(H^0(K_n; \mathbf{Z}), n) \times K(H^n(K_n; \mathbf{Z}), 0) \simeq K_n \times \text{Hom}(\mathbf{Z}, \mathbf{Z})$$

(here of course  $H^n(K_n; \mathbf{Z}) = \text{Hom}(H_n(K_n; \mathbf{Z}), \mathbf{Z}) = \text{Hom}(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}$ .) The equivalence above can be explicitly constructed as follows. We pointed out earlier that  $K(\mathbf{Z}, n) \simeq SP^\infty(S^n, *)$  (this equivalence can be seen in many ways; cf. [DT] or [M]) and the abelian monoid structure on  $K_n = K(\mathbf{Z}, n)$  is induced from the symmetric product pairing (which we write additively). Given a map  $f : \mathbf{Z} \longrightarrow \mathbf{Z}$  determined by an integer  $k$ , we can consider the  $k$ -fold map  $S^n \longrightarrow S^n$  and extend it out (additively) to a map  $(k) : SP^\infty(S^n, *) \rightarrow SP^\infty(S^n, *)$  and hence to an element  $(k) \in \text{Map}(K_n, K_n)$ . On the other hand,  $K_n$  maps to the translation elements in  $\text{Map}(K_n, K_n)$  and the product map  $(x, k) \mapsto T_x + (k)$  induces the equivalence  $K_n \times \text{Hom}(\mathbf{Z}, \mathbf{Z}) \rightarrow \text{Map}(K_n, K_n)$ . The homotopy inverse sends  $f \in \text{Map}(K_n, K_n)$  to  $(f(x_0), \deg f)$  where  $x_0 \in K_n$  is the basepoint and  $\deg f$  is the degree of the induced map at the level of  $\pi_n$ .

Note at this point that since  $H_n(S^n) \cong \pi_n(SP^\infty(S^n, *))$ , the map  $(k)$  induces multiplication by  $k$  at the level of  $\pi_n$  and so  $(k)$  is a homotopy equivalence if and only if  $k = \pm 1$ , in which case multiplication by  $k$  is in  $\text{Aut}(\mathbf{Z})$ . Notice also that an element in  $K_n$  acting by translation can be homotoped to the identity and hence the map  $T : K_n \rightarrow \text{Map}(K_n, K_n)$  factors through  $\text{Aut}(K_n)$ . These two facts put together show that the diagram in 10.2 commutes. It remains to show that the left vertical map is an equivalence but it is not hard to see that the right-hand equivalence we just described restricts to  $\text{Aut}(K_n)$  and the proposition follows. ■

**Remark 10.3:** We can replace  $\mathbf{Z}$  by any abelian group  $G$  in 10.2 above and prove similarly that  $\text{Aut}(K(G, n)) \simeq K(G, n) \times \text{Aut}(G)$ . At the level of simplicial groups,  $\text{Aut}(K(G, n))$  is given as a semi-direct product of  $\text{Aut}(G)$  and  $K(G, n)$  (May). When  $G = \mathbf{Z}$ ,  $\text{Aut}(\mathbf{Z}) \cong \mathbf{Z}_2$  and  $\text{Aut}(K_n)$  consists of two copies of  $K_n$  (consisting resp. of “orientation” preserving and orientation reversing homotopy equivalences).

**Theorem 10.4:** *Let  $M \in \underline{\mathcal{C}}_n$ . Then the bundle  $K(\mathbf{Z}, n) \rightarrow E_{SP^\infty} \rightarrow M$  is trivial if and only if  $M$  is oriented.*

PROOF: The bundle  $E_{SP^\infty}$  is classified by a map  $M \longrightarrow B\text{Aut}(K(\mathbf{Z}, n))$  and at the level of spaces we get a (trivial) fibration

$$K(\mathbf{Z}, n+1) \longrightarrow B(\text{Aut}(K(\mathbf{Z}, n))) \longrightarrow B(\text{Aut}(\mathbf{Z})).$$

The classifying map  $f : M \longrightarrow B\text{Aut}(K(\mathbf{Z}, n))$  lifts to  $K(\mathbf{Z}, n+1)$  if and only if the composite  $M \rightarrow B(\text{Aut}(\mathbf{Z}))$  is null homotopic or equivalently if the induced map  $\phi : \pi_1(M) \longrightarrow \text{Aut}(\mathbf{Z})$  is

trivial. The action of  $\pi_1(M)$  on  $\mathbf{Z}$  described by the map  $\phi$  corresponds to the action of  $\pi_1(M)$  on  $\mathbf{Z} = \pi_n(K(\mathbf{Z}, n))$  in the bundle in 10.4 (this follows directly from the many facts stated in the proof of 10.2). But  $M$  being oriented, the tangent bundle  $\tau M$  (and hence its compactified counterpart  $\widehat{\tau M}$ ) is trivial over the 1-skeleton. Consequently,  $E_{SP^\infty}$  restricted to the one skeleton of  $M$  is also trivial and so is the action of  $\pi_1(M)$  on the fiber. Namely,  $\pi_1(M)$  acts trivially on  $\pi_n(K(\mathbf{Z}, n)) = \mathbf{Z}$  and as indicated above the map  $f$  must lift to a map  $\tilde{f} : M \longrightarrow K(\mathbf{Z}, n+1)$ . Since  $M$  is  $n$  dimensional,  $\tilde{f}$  is null-homotopic and  $E_{SP^\infty}$  is trivial.

To prove the other easier direction, suppose  $E_{SP^\infty}$  is trivial that is  $E_{SP^\infty} \simeq K(\mathbf{Z}, n) \times M$ . The inclusion  $\widehat{\tau M} \subset E_{SP^\infty}$  composed with projection yields a map of  $\widehat{\tau M} \rightarrow K(\mathbf{Z}, n)$  and hence a Thom class in  $H^n(\widehat{\tau M}; \mathbf{Z})$ . This is equivalent to giving an orientation class for  $M$  and the proposition follows. ■

**Theorem 10.5:** *Let  $N \hookrightarrow M$  be a closed ANR of a closed, oriented manifold  $M \in \underline{\mathcal{C}}_n, n \geq 2$ . Then*

$$SP^\infty(M - N, *) \xrightarrow{\simeq} \text{Map}_0(M, N, SP^\infty(S^n, *)).$$

PROOF: Here of course and since  $E_{SP^\infty(M-N)}$  is trivial, the space of sections and the space of maps into the fiber coincide. The homotopy equivalence is a consequence of 1.3 (or 9.13.) ■

**Remark 10.6:** When  $M$  is parallelizable, the map  $S$  has the following alternate description. Start with  $M$  compact and for each  $x \in M$  choose an open ball  $D_x \subset M$  containing  $x$  and such that  $D_x/\partial D_x \simeq S^n$  canonically. The quotient maps  $M \longrightarrow D_x/\partial D_x = M/M - D_x \simeq S^n$  give rise to maps

$$s_x : M \longrightarrow S^n \hookrightarrow SP^\infty(S^n, *), \quad \forall x \in M$$

and hence to a correspondence  $s : M \longrightarrow \text{Map}(M, SP^\infty(S^n, *))$  which extends additively to

$$\bar{s} : SP^\infty(M) \longrightarrow \text{Map}(M, SP^\infty(S^n, *)).$$

It isn't hard to see that  $\bar{s} \simeq S$  (Another variation on this construction is given in §12.)

A direct consequence of proposition 2.7 and from the fact that  $SP^\infty(-)$  is a homotopy functor, it follows that  $\pi_*(SP^\infty(-))$  defines a homology theory and a well-known theorem of Dold and Thom identifies it with ordinary singular homology theory; i.e.

$$10.7 \quad SP^\infty(X, *) = \prod_i K(\tilde{H}_i, (X; \mathbf{Z}), i)$$

Combining 10.1 with 10.7 we get the equivalence

$$\prod_i K(\tilde{H}_i, (M - N; \mathbf{Z}), i) \simeq \prod_{1 \leq i \leq n} K(H^{n-i}(M/N, \mathbf{Z}), i)$$

from which we easily deduce our main application

**Corollary 10.8:** (Alexander-Poincaré Duality) *Let  $N \hookrightarrow M$  be a closed ANR in an orientable manifold  $M$  of dimension  $n$ . Then  $\tilde{H}_i(M - N; \mathbf{Z}) \cong H^{n-i}(M, N; \mathbf{Z})$ .*

Similarly, considering the equivalence  $SP^\infty(M, *) \simeq \text{Map}_c(M, \partial M, K(\mathbf{Z}, n))$  for  $M$  compact with boundary yields

**Corollary 10.9** (Lefschetz-Poincaré Duality) *Let  $M$  be compact with boundary, of dimension  $n$ , and suppose  $\text{int}M$  is orientable. Then  $H_q(M) \cong H^{n-q}(M, \partial M)$ .*

**Example 10.10:** The classical Alexander duality is stated as follows. Let  $X$  be a finite complex embedded in  $S^n$  ( $n \geq 1$ ). By 10.8 we have that  $H^{n-i}(S^n, X) \cong \tilde{H}_i(S^n - X)$  and the relative sequence for the pair  $(X, S^n)$  shows that

$$\tilde{H}_i(S^n - X) \cong H^{n-1-i}(X; \mathbf{Z}).$$

When  $i$  corresponds to one less the “codimension” of  $X$  in  $S^n$ , the isomorphism above has a very nice geometric interpretation (see [KT] for example). Suppose  $X = M$  is a smooth closed  $m$  manifold embedded in  $S^n$ . It has a unit normal sphere bundle  $S^{n-m-1} \rightarrow \nu(M) \rightarrow M$  and the homology class of the fiber in  $\tilde{H}_{n-m-i}(S^n - X)$  is dual to the cohomology orientation class in  $H^m(M; \mathbf{Z})$ . Note in this case that the class in  $\tilde{H}_{n-m-i}(S^n - X)$  is spherical.

## §11 Applications

**§11.1 On Theorems of McDuff and Segal:** As pointed out in the introduction, the configuration space functor  $C^\infty$  has been studied in [S2] and [McD1] where special versions of theorem 1.1 have been proved. In this subsection, we extend their results in several directions.

Consider the subspace of  $C^{(k)}(M) \subset C(M)^k$  consisting of tuples of configurations which are pairwise disjoint. More explicitly

$$C^{(k)}(M) = \{(\zeta_1, \dots, \zeta_k) \in C(M)^k \mid \zeta_i \cap \zeta_j = \emptyset, i \neq j\}.$$

It is direct to see that  $C^{(k)}(M)$  is a particle space and hence for parallelizable pairs  $(M, N)$  we have (theorem 9.15)

$$H_*(C^{(k)}(M - N); \mathbf{Z}) \xrightarrow{S_*} H_*(\text{Map}(M, N, C^{(k)}(S^n, *)); \mathbf{Z})$$

**Lemma 11.1:** *Let  $\bigvee^k S^n$  denote the  $k$ -th wedge,  $n \geq 1$ . Then  $C^{(k)}(S^n, *) \simeq \bigvee^k S^n$ .*

PROOF: As in [S1], we let  $C_\epsilon^{(k)}(S^n, *)$  be the open set of  $C^{(k)}(S^n, *)$  consisting of multiconfigurations  $(\zeta_1, \dots, \zeta_k)$  such that at least  $k - 1$  such particles are disjoint from the closed disk  $U_\epsilon$  of radius  $\epsilon > 0$  about the south pole  $*$ . Notice that there is a radial homotopy, injective on the interior of  $U_\epsilon$  that expands the north cap  $U_\epsilon$  over the sphere and takes  $\partial U_\epsilon$  to  $*$ . Such an expansion retracts  $C_\epsilon^{(k)}(S^n, *)$  to the wedge product  $C(S^n, *) \vee \dots \vee C(S^n, *)$ . Now since  $C^{(k)}(S^n, *)$  is the union of the  $C_\epsilon^{(k)}(S^n, *)$  for  $\epsilon > 0$ , we get that  $C^{(k)}(S^n, *) \simeq \bigvee^k C(S^n, *)$ .

It remains to show that  $C(S^n, *) \simeq S^n$ . Here too we consider the subspace

$$C_\epsilon(S^n, *) = \{D \in C(S^n, *) \mid D \cap U_\epsilon = \{\text{at most one point}\}\}$$

where  $U_\epsilon$  is an epsilon neighborhood of the north pole (again the south pole corresponds to  $*$ ). Then radial expansion of  $U_\epsilon$  ( $N$  is fixed) maps  $(U_\epsilon, \partial U_\epsilon)$  to  $(S^n, *)$  (and is injective on  $U_\epsilon$  hence extending to  $C$ ). The one point configurations in  $U_\epsilon$  now produce a homeomorphism  $C_\epsilon(S^n, *) \simeq S^n$  and since again  $C(S^n, *) = \bigcup_\epsilon C_\epsilon(S^n, *)$  the lemma follows. ■

**Proposition 11.2:** *Let  $M \in \mathcal{C}_n$  be a closed manifold and  $N \subset M$  such that  $(M, N)$  is parallelizable. Then*

$$S_* : H_*(C^{(k)}(M - N)) \cong H_*(\text{Map}(M, N, \bigvee^k S^n)).$$

When  $M - N = \mathbf{R}^n \cong D^n$ ,  $D^n$  here is the closed unit disc, then components of  $C^{(k)}(\mathbf{R}^n)$  can be identified with the direct limit  $C^\infty(D^n, p)$  constructed in 7.5. We have

**Corollary 11.3:** *The scanning map  $S : C^{(k)}(\mathbf{R}^n) \longrightarrow \Omega^n(\bigvee^k S^n)$  induces a homology isomorphism. When  $k = 1$  we recover the following classical result of Segal [S2]*

$$E_* : H_*(C^\infty(D^n, p); \mathbf{Z}) \cong H_*\left(\lim_i C_i(\mathbf{R}^n); \mathbf{Z}\right) \xrightarrow{\cong} H_*(\Omega_0^n S^n; \mathbf{Z}).$$

**Example 11.4:** It can be checked (exactly as in 11.1) that  $DDiv^k(S^n, *) \simeq \bigvee^k K(\mathbf{Z}, n)$  and that the following commutes (up to homotopy)

$$\begin{array}{ccc} C^{(k)}(M - N) & \xrightarrow{S} & \text{Map}(M, N, \bigvee^k S^n) \\ \downarrow \subset & & \downarrow \\ DDiv^k(M - N) & \xrightarrow{\simeq} & \text{Map}(M, N, \bigvee^k K(\mathbf{Z}, n)) \end{array}$$

where  $M$  and  $N$  are as in the statement of theorem 1.1. We quickly remind the reader that  $DDiv^k(M)$  is the set of  $k$ -tuples of positive divisors which are pairwise disjoint. We finally point out that the right vertical map in the diagram is induced from the inclusion  $S^n \hookrightarrow K(\mathbf{Z}, n)$  and the homotopy equivalence at the bottom follows from the fact that  $\pi_1(DDiv^k(\mathbf{R}^n))$  is abelian (which is left for check to the reader). ■

**Example 11.5:** (*Spaces of positive and negative particles*) [McD1] also introduces the functor  $C^\pm$  discussed in §1. This is given as the quotient of  $C(M) \times C(M)$  with the relation

$$(\langle x, x_1, \dots, x_n \rangle, \langle x, y_1, \dots, y_m \rangle) \sim_R (\langle x_1, \dots, x_n \rangle, \langle y_1, \dots, y_m \rangle).$$

One can show that  $C^\pm$  abelianizes fundamental group and hence theorem 1.3 applies. Observe that since  $C(S^n, *) \simeq S^n$ , it follows that  $C^\pm(S^n, *) \simeq S^n \times S^n / \Delta$  where  $\Delta(S^n)$  is the diagonal copy of  $S^n$  in  $S^n \times S^n$ . The following homotopy equivalence is a special case of [McD1] or of theorem 1.3

$$C^\pm(\mathbf{R}^n) \simeq \Omega^n(S^n \times S^n / \Delta(S^n)).$$

**§11.2 Symmetric products with bounded multiplicities:** In this subsection we prove theorem 1.5 in the introduction. Recall that  $SP_d^\infty$  was defined as the particle functor of the first kind

$$SP_d^\infty(M) = \{\sum n_i x_i \in SP^\infty(M) \mid n_i \leq d\}.$$

We first need the following analog of 11.1.

**Lemma 11.6:** *There is a homotopy equivalence  $SP_d^\infty(S^k, *) \simeq SP^d(S^k)$ .*

PROOF: Let  $*$   $\in S^k$  and  $U_\epsilon$  be as in 11.1, and let  $W_\epsilon$  be the subspace consisting of  $\langle x_1, x_2, \dots, x_n \rangle \in SP_d^\infty(S^k, *)$  such that at most  $d$  points in the tuple lie inside  $U_\epsilon$ . By definition of  $SP_d^\infty(S^k, *)$  each of its elements must fall into a  $W_\epsilon$  for some  $\epsilon$  and hence

$$SP_d^\infty(S^k, *) \simeq \bigcup_{\epsilon} W_\epsilon.$$

Now using the radial retraction of 11.1, it is clear that each  $W_\epsilon$  is homotopically  $SP^d(S^k)$  (since by taking a configuration  $\langle x_1, x_2, \dots, x_n \rangle$  and shrinking (at least)  $n - d$  points of it to basepoint  $*$ , we end up in  $SP^d(S^k)$ .) The lemma follows. ■

**Theorem 11.7:** *Let  $M$  and  $N$  be as in 1.1. Then*

$$S : SP_d(M - N) \longrightarrow \text{Map}(M, N \cup \partial M, SP^d(S^n))$$

*is a homotopy equivalence whenever  $d > 1$  and a homology equivalence when  $d = 1$ .*

PROOF: Let  $X = M - N$ . The claim amounts to showing that  $\pi_1(SP_d^n(X))$  is abelian when  $n > 1$  and  $d > 1$ . We know already (9.13) that  $\pi_1(SP^n(X))$  is abelian for  $n > 1$ . Since  $H_1(SP^n(X); \mathbf{Z}) \cong H_1(SP^{n+1}(X); \mathbf{Z})$ , it follows that the inclusion  $SP^2(X) \hookrightarrow SP^n(X)$  for  $n \geq 2$  induces an isomorphism in fundamental group. Consider at this point the commutative diagram

$$\begin{array}{ccc} SP_d^2(X) & \hookrightarrow & SP^2(X) \\ \downarrow \subset & & \downarrow \subset \\ SP_d^n(X) & \hookrightarrow & SP^n(X) \end{array}$$

Any element  $\alpha \in \pi_1(SP_d^n(X))$  factors through the subset  $SP_d^2(X)$  in  $SP^n(X)$ . But for  $d > 1$ , these last two spaces coincide and since  $\pi_1(SP^2(X))$  is abelian, the claim follows. ■

**COROLLARY 11.8:** Restricting to the case  $M = D^n$  the closed unit disc,  $N = \emptyset$ , we recover 1.5 in the introduction (see also 5.2). When  $X = \mathbf{C} \cong \mathbf{R}^2$ , the space  $SP_d^n(\mathbf{C})$  can be identified with the space of monic polynomials  $p$  of degree  $n$ ,  $p(z) = (z - x_1) \cdots (z - x_n)$  such that  $p$  has no roots of multiplicity greater than  $d$ . Notice in this case that  $SP^d(S^2)$  is diffeomorphic to the  $d$ th complex projective space  $\mathbf{P}^d$  and hence we obtain the following corollary also proved in [GKY]

**Corollary 11.9 [GKY]:** *There is a correspondence*

$$\left\{ \begin{array}{l} \text{Monic complex polynomials of degree } n \\ \text{and roots of multiplicity } d > 1 \end{array} \right\} \longrightarrow \Omega_0^2 \mathbf{P}^d$$



which is a homotopy equivalence in the direct limit when  $n \longrightarrow \infty$ .

**Remark 11.10:** One can prove more in this case (as [GKY] do) by showing that the correspondence above is a homotopy equivalence through a range. This is a good place to point out that our main theorem 1.1 is quite likely to have an unstable version which would state that scanning  $S$  is a homology equivalence through a range increasing with the multidegree of the *Par* spaces.

**§11.3 Rational curves on toric varieties and a theorem of Guest:** A toric variety  $V$  is a projective variety that can be defined by equations of the form “monomial in  $z_0, \dots, z_n$  = monomial in  $z_0, \dots, z_n$ ”. As an example, consider the quartic

$$M_2 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{P}^3 \mid z_2^2 = z_1 z_3\}.$$

A rational curve on  $V$  is a holomorphic image of  $\mathbf{P}^1 = S^2$  in  $V$  and one is interested in studying the space of all such curves. The interest here stems from the relevance of these spaces to problems in Gauge theory, Sigma models for physicists and even Control theory for engineers (cf. [BHMM], [C<sup>2</sup>M<sup>2</sup>] and [S1] for a general discussion of the subject).

We denote by  $\text{Hol}(\mathbf{P}^1, V)$  the space of all holomorphic maps from  $\mathbf{P}^1$  into  $V$ . As is customary, the study of this space proceeds by first restricting attention to the subspace of based maps (which consists of maps that fix a given basepoint.) Choosing  $x_0 \in \mathbf{P}^1$  and  $* \in V$ , we let  $\text{Hol}^*(\mathbf{P}^1, V)$  be the subspace of  $f : \mathbf{P}^1 \rightarrow V$  such that  $f(x_0) = *$ . It has to be pointed out that the topology of  $\text{Hol}^*(\mathbf{P}^1, V)$  could vary with the choice of the basepoint  $*$  (unless for example  $V$  is homogeneous).

It turns out that for a generic choice of a basepoint  $* \in V$ , a map  $f \in \text{Hol}^*(\mathbf{P}^1, V)$  admits a representation by polynomials. More precisely, given  $f : \mathbf{P}^1 \longrightarrow V$  holomorphic, the composite

$$\mathbf{P}^1 \xrightarrow{f} V \hookrightarrow \mathbf{P}^n \quad (\text{for some } n)$$

is also holomorphic and so  $f$  can be represented by the map  $[p_0(z) : \dots : p_n(z)]$  where the  $p_i(z)$  satisfy the same set of equations as  $V$  and of course have no roots in common. Notice also that when  $f$  is basepoint preserving, the  $p_i$  can be chosen to be monic (and hence are uniquely determined).

**Example 11.11 (Guest):** Consider the quadric curve  $M_2$  described earlier. It can be seen that  $M_2$  is smooth but at the one point  $[1 : 0 : 0 : 0]$ . A (based) rational curve  $f : S^2 \rightarrow M_2$ , sending the north pole to any point other than this singular point, has a representation in terms of a 4-tuple of polynomials  $(q_1, q_2, q_3, q_4)$  that are coprime, monic and satisfying the equation  $q_3^2 = q_2 q_4$ . The map  $f$  is therefore equivalent to the choice of four monic polynomials  $p_1, p_2, p_3$  and  $p_4$  such that

$$\begin{aligned} (q_1, q_2, q_3, q_4) &= (p_4, p_1^2 p_2, p_1 p_2 p_3, p_2 p_3^2) \\ (p_1, p_3) &= 1, (p_2, p_4) = 1 \\ \deg p_4 &= \deg p_1^2 p_2 = \deg p_1 p_2 p_3 = \deg p_2 p_3^2 = d \end{aligned}$$

where  $d$  is the degree of  $f$ . This last representation can be reformulated in terms of divisors  $D_1, D_2, D_3$  and  $D_4$  given by the roots of the  $q_i$  and hence satisfying

$$\begin{aligned} D_1 \cap D_2 \cap D_3 \cap D_4 &= \emptyset, \deg D_i = d \\ D_2, D_3, D_4 &\text{ are of the form } \eta + 2\phi, \eta + \phi + \chi, \eta + 2\chi \text{ where } \phi \cap \chi = \emptyset. \end{aligned}$$

Generally, the  $p_i$ 's one associates to  $f \in \text{Hol}^*(\mathbf{P}^1, V)$  being monic, their root data totally determine the map  $f$ . For a general toric variety  $V$ , which we assume to be non-singular (the singular case is a little more intricate but can still be treated analogously), a rational map  $f : \mathbf{P}^1 \longrightarrow V$  will have a multidegree  $D$  associated to it where

$$D = (d_1, \dots, d_p) \in \pi_2(V) \cong \bigoplus_{i=1}^p \mathbf{Z}$$

and this multidegree parametrizes components of  $\text{Hol}^*(\mathbf{P}^1, V)$ . We say  $D \rightarrow \infty$  if all the components  $d_i$  tend to infinity.

**Lemma 11.12:** *There is a homeomorphism  $\text{Hol}_D^*(S^2, V) \cong \text{Par}_D(S^2 - \infty)$  for some particle space  $\text{Par}^\infty(S^2 - *)$ , sending  $f \in \text{Hol}_d^*(S^2, V)$  to the roots of the  $p_i(z)$ ,  $0 \leq i \leq n$  in its polynomial representation.*

PROOF: The proof is direct since if two polynomial representations given by  $p_i$  and  $p'_i$ ,  $1 \leq i \leq n$ , have root data lying in disjoint sets, then their products  $p_i p'_i$  will give rise to another representation describing a new holomorphic map  $S^2 \rightarrow V$ . ■

We can up to homeomorphism construct stabilization maps

$$11.13 \quad \text{Hol}_D^*(S^2, V) \longrightarrow \text{Hol}_{D+D'}^*(S^2, V)$$

as in §7. This induces stabilization maps at the level of  $\text{Par}_D^\infty(S^2 - \infty)$  and the direct limit is a component of  $\text{Par}(S^2 - \infty)$  (see §5).

**Theorem 11.14:** (Guest) *Let  $X$  be a projective toric variety (non-singular). The inclusions  $i_D : \text{Hol}_D^*(S^2, V) \longrightarrow \Omega_D^2 V$  induce a homotopy equivalence when  $D$  goes to  $\infty$ ; i.e.*

$$\lim_{D \rightarrow \infty} \text{Hol}_D^*(S^2, V) \xrightarrow{\lim i_D} \Omega_0^2 V$$

where  $\Omega_0^2 V$  is any component.

PROOF: Arguments of Segal and Guest show that in this general case scanning and the inclusion  $i$  fits in a homotopy commutative diagram

$$\begin{array}{ccc} \text{Hol}_D^*(S^2, V) & \xrightarrow{i_D} & \text{Map}_D^*(S^2, V) \\ \downarrow & & \downarrow \simeq \\ \text{Par}_c(S^2 - \infty) & \xrightarrow{S} & \text{Map}_c^*(S^2, \text{Par}^\infty(S^2, *)). \end{array}$$

where from above the map  $\text{Hol}_D^*(S^2, V) \rightarrow \text{Par}_c(S^2 - \infty)$  can be identified with the map of  $\text{Hol}_D^*$  into the direct limit of the system in 11.13 (note that  $\text{Map}_c^*$  denotes any component of  $\text{Map}^*(S^2, \text{Par}^\infty(S^2, *))$  and they're all homotopy equivalent by 9.14). The scanning map  $S$  at the bottom will be a homotopy equivalence according to 1.3 if we can show that  $\pi_1(\text{Par}_c(S^2 - \infty))$  is abelian. It is shown in ([BHMM], corollary 9.9) that  $\pi_1(\text{Hol}_D^*(S^2, V))$  is abelian for  $D$  consisting of

multidegrees  $(d_1, \dots, d_p)$  with  $d_i \geq 2$ . Moreover for  $D$  and  $D'$  with this property,  $\pi_1(\text{Hol}_D^*(S^2, V)) \cong \pi_1(\text{Hol}_{D'}^*(S^2, V))$  hence implying that in the direct limit  $\pi_1(\text{Hol}^*(S^2, V))$  is well defined and abelian. The claim now follows.  $\blacksquare$

## §12 Spanier-Whitehead Duality

The ideas of the previous sections can be adapted to prove Spanier-Whitehead duality for general homology theories  $h_*$  and for any finite type CW complex  $X$ . The material below is known in some form or another and we include it in this section for completeness.

As a start we denote by  $CW$  the category of connected finite CW complexes. For a given  $X \in CW$ , we let  $D(X, k)$  be its Spanier-Whitehead dual (or  $S$ -dual). An  $S$ -dual always comes equipped with a map  $X \wedge D(X, k) \longrightarrow S^k$  (see [CM]).

One can construct the  $S$ -dual of any  $X \in CW$  very concretely. Indeed, since  $X$  is finite, it embeds in some big enough sphere  $S^k$ . The complement  $Y = S^k - X$  can now be chosen as a spanier-whitehead dual for  $\Sigma X$ ; i.e.  $Y = D(\Sigma X, k)$  ([CM]). The  $S$ -dual of  $X$  can then be taken to be  $\Sigma D(\Sigma X, k)$ . It follows for instance that the  $S$  dual of  $S^n$  is  $S^{k-n}$ .

Given a connective  $\Omega$  spectrum  $\mathbf{E} = \{E_i, i = 1, \dots\}$ , we have that

$$E_0 = \lim_m \Omega^m E_m \equiv \Omega^\infty \mathbf{E}$$

and more generally  $E_n = \Omega^\infty(S^n \wedge \mathbf{E})$ . The functor  $\Omega^\infty$  is a functor from spectra to spaces.

We can associate to  $\mathbf{E}$  the functor  $F_{\mathbf{E}}$  defined as follows

$$F_{\mathbf{E}} : X \mapsto F(X) = \Omega^\infty(\mathbf{E} \wedge X).$$

Notice that by definition  $F_{\mathbf{E}}(S^n) = E_n$ . Notice also that

$$\begin{aligned} \pi_i(F_{\mathbf{E}}(X)) &= [S^i, \Omega^\infty(X \wedge \mathbf{E})] = \lim_n [S^i, \Omega^n(E_n \wedge X)] \\ &= \lim_n \pi_{i+n}(E_n \wedge X) = h_i(X) \end{aligned}$$

where  $h_*$  is the generalized homology theory associated to  $\mathbf{E}$ .

**Remark 12.1:** Generally, given a spectrum  $\mathbf{E}$ , we denote by  $\Omega^\infty$  the functor obtained as the composite of the functor which converts any spectrum into an equivalent  $\Omega$  spectrum  $\mathbf{E}$  followed by the functor which passes from  $\mathbf{E}$  into the space  $E_0$  (see [Ad], p:22). Notice that  $E_0$  doesn't generally correspond to  $E'_0$  (as the sphere spectrum  $\mathbf{S}^0$  does illustrate already).

**Theorem 12.2:** Let  $F = F_{\mathbf{E}}$  for some spectrum  $\mathbf{E}$ . Then  $\forall X \in CW$ , there is a homotopy equivalence

$$S : F(X) \xrightarrow{\simeq} \text{Map}^*(D(X, k), F(S^k)).$$

PROOF: Let  $X$  be a finite CW complex. Then  $X \subset S^k$  for some  $k$  and  $D(\Sigma X, k) = S^k - \Sigma X$ . Since  $X$  and  $D(\Sigma X, k)$  are disjoint, we can consider the map

$$\hat{S} : X \times D(\Sigma X, k) \longrightarrow S^{k-1}, (x, y) \mapsto \frac{x - y}{|x - y|} \in S^k.$$

We can assume  $X$  to be embedded in the positive quadrant in  $\mathbf{R}^n \subset S^k$  with the point at infinity  $\infty \in S^k$  adjoined. This means that  $\hat{S}(\infty, y) = 1, \forall y \in D(\Sigma X, k)$ . On the other hand and since  $X$  is compact, it lies in a ball  $B \in S^k$ . Choose a point  $p \in D(\Sigma X, k)$  which is not in  $B$ . The map  $\hat{S}|_{X \times p}$  extends to  $B \times p$  and since  $B$  is contractible we get an extension

$$\hat{S} : X \times D(\Sigma X, k) \cup c(X \times p) \longrightarrow S^{k-1}$$

where  $c$  denotes the cone construction. It then follows that up to homotopy, the map  $\hat{S}$  gives rise to the map

$$X \wedge D(\Sigma X, k) \longrightarrow S^{k-1}.$$

Suspending both sides yields a map  $X \wedge D(X, k) \longrightarrow S^k$  and hence by adjoining a map

$$12.3 \quad \hat{S} : X \longrightarrow \text{Map}^*(D(X, k), S^k)$$

where the mapping space on the right is pointed. Of course we can compose with the map  $i : \text{Map}^*(D(X, k), S^k) \rightarrow \text{Map}^*(D(X, k), F(S^k))$  induced from the “identity”  $S^k \rightarrow F(S^k)$ . Since  $F$  is an infinite loop functor, 12.3 composed with  $i$  extends to the desired map

$$S : F(X) \longrightarrow \text{Map}^*(D(X, k), F(S^k)).$$

We show that  $S$  is a homotopy equivalence by inducting on cells of  $X$ . Let  $X^{(i)}$  denote the  $i$ -th skeleton of  $X$  and consider the standard cofibration  $X^{(i-1)} \hookrightarrow X^{(i)} \longrightarrow \bigvee S^i$ . Applying  $\text{Map}^*(-, S^k)$  yields a fibration sequence and a homotopy commutative diagram

$$\begin{array}{ccc} \prod F(S^{i-1}) & \xrightarrow{\simeq} & \prod \Omega^{k-i+1} F(S^k) \\ \downarrow & & \downarrow \\ F(X^{(i-1)}) & \xrightarrow{S} & \text{Map}^*(D(X^{(i-1)}, k), F(S^k)) \\ \downarrow & & \downarrow \\ F(X^{(i)}) & \xrightarrow{S} & \text{Map}^*(D(X^{(i)}, k), F(S^k)) \end{array}$$

The left hand vertical sequence is a quasifibration since  $F$  is a homology theory. The top horizontal map is an equivalence since  $\Omega F(S^l) \simeq F(S^{l-1})$  while the bottom map is an equivalence by induction. This then implies that the middle map  $S$  is also an equivalence and the proof follows. ■

EXAMPLE 12.4: By Dold-Thom, we know that  $SP^\infty(-)$  is associated to the Eilenberg-MacLane spectrum  $K(\mathbf{Z})$  (i.e.  $\pi_*(SP^\infty(-)) \cong H_*(-; \mathbf{Z})$ ), while the functor  $Q(-)$  given by

$$QX = \Omega^\infty \Sigma^\infty(X)$$

is known to be associated to the sphere spectrum (i.e.  $\pi_*(Q(X)) = \pi_*^S(X)$ ). One has then the following homotopy equivalence (described in [C])

$$Q(X) \xrightarrow{\simeq} \text{Map}^*(D(X, k), QS^k).$$

At this point, consider  $A, B \in S^k$  for some large  $k$ . Recall that  $A$  is  $n$  dual to  $B$  if  $A \cap B = \emptyset$  and each is a strong deformation retract of the complement of the other.

**Corollary 12.5:** (Spanier-Whitehead duality) *Let  $\mathbf{E}$  be a connective spectrum and let  $h$  be the homology theory defined by  $\mathbf{E}$ ; i.e.  $h_*(X) = [S^0, E \wedge X]$ . Suppose  $A, B \in S^k$ ,  $A$  and  $B$  are  $n$  dual. Then there is an isomorphism*

$$h_i(B) \cong h^{n-1-i}(A).$$

PROOF: Let  $\mathbf{E}$  be a connective spectrum with a unit. We can choose  $\mathbf{E}$  to be an  $\Omega$  spectrum. Indeed if it weren't such, then the spectrum representing the generalized homology theory defined by  $\mathbf{E}$  still is. And so as far as homology is involved, we could have chosen  $\mathbf{E}$  to be an  $\Omega$  spectrum to start with.

Theorem 12.3 now shows that  $F_{\mathbf{E}}(X) \simeq \text{Map}(D(X, k), F_{\mathbf{E}}(S^k))$  and it follows that

$$\begin{aligned} h_i(X) &= \pi_i(F_{\mathbf{E}}(X)) = \pi_i(\text{Map}(D(X, k), F_{\mathbf{E}}(S^k))) \\ &= [S^i \wedge D(X, k), F_{\mathbf{E}}(S^k)] = [D(X, k), \Omega^i E_k] \\ &= [D(X, k), E_{k-i}] = h^{k-i}(D(X, k)). \end{aligned}$$

Here we used the facts that  $h_i(X) = \pi_i(F_{\mathbf{E}}(X))$  and  $F_{\mathbf{E}}(S^n) \simeq E_n$ . This concludes the proof. ■

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